Identity and Aboutness*

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Abstract

This paper develops a theory of propositional identity which distinguishes necessarily equivalent propositions that differ in subject-matter. Rather than forming a Boolean lattice as in extensional and intensional semantic theories, the space of propositions forms a non-interlaced bilattice. After motivating a departure from tradition by way of a number of plausible principles for subject-matter, I will provide a Finean state semantics for a novel theory of propositions, presenting arguments against the convexity and nonvacuity constraints which Fine (2016, 2017a,b) introduces. I will then move to compare the resulting logic of propositional identity (PI^1) with Correia's (2016) logic of generalised identity (GI), as well as the first degree fragment of Angell's (1989) logic of analytic containment (AC). The paper concludes by extending PI^1 to include axioms and rules for a subject-matter operator, providing a much broader theory of subject-matter than the collection of principles with which I will begin.

Keywords Identity · Subject-matter · Hyperintensionality · State Semantics

1 Intensionalism

When do two sentences express the same proposition in virtue of their logical form? It is important to stress that the present understanding of propositions is intended to be worldly: propositions are, so to speak, *things being a certain way*, rather than mere representations of things being some way or other. *Intensionalism* is the view that what it is for propositions to be identical is for them to be necessarily equivalent. For instance, reading $\ulcorner\varphi \equiv \psi\urcorner$ as ⌜For it to be the case that φ *just is* for it to be the case that ψ⌝, Rayo (2013, p. 66) writes, "'$\phi \equiv \psi$' should be thought of as equivalent to '$\Box(\phi \leftrightarrow \psi)$'." Part of what makes intensionalism appealing is that a simple and strong range of necessary equivalences may be derived in any normal modal logic, shedding light on the nature of propositional identity and the abstract structure of the space of propositions. Intensionalism is to be contrasted with the claim that necessary equivalence may serve to approximate propositional identity despite failing to be co-extensive with propositional identity, or else given some

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specific application, that necessary equivalence may prove to be of some utility. Certainly it should be admitted that intensional theories of propositions have been of great utility in philosophy, logic, linguistics, and computer science. Nevertheless, the present inquiry concerns the nature of propositional identity itself, and not some approximation or useful application.

Instead of defining propositional identity as the intensionalist does, or in yet some further way, *primitivism* claims that propositional identity is conceptually basic, and so an informative definition in other terms cannot be provided. Accordingly, primitivists cannot follow intensionalists in employing a modal logic to derive a range of theorems for propositional identity from its definition. In an attempt to characterise propositional identity, a primitivist is forced to make a fresh start by axiomatising propositional identity rather than defining propositional identity in terms of other primitive notions. Of course, free from all constraints, little progress can be made, leaving one with no more than suspicions about which identities hold in full generality. For instance, is A the same or different proposition as either $A \wedge A$ or $A \vee A$? What about A and either of the propositions $A \vee (A \wedge B)$ or $A \wedge (A \vee B)$?

Even if propositional identity cannot be defined in other terms, a primitivist must nevertheless look to some prior conception of the theoretical role that propositions are meant to play in order to guide the ambition to axiomatise propositional identity. For instance, suppose that: (1) the proposition expressed by a sentence on a given interpretation is identified with that sentence's truth-condition; where (2) the truth-condition for an interpreted sentence is taken to be the set of possible worlds in which that sentence is true.[1] Leaving the interpretation implicit, it follows that sentences which are true in the same possible worlds express the same proposition, where of course sentences which express the same proposition are true in the same possible worlds, and so propositional identity ends up co-extensive with necessary equivalence. In particular, $\ulcorner A \urcorner$ is true in the same worlds as $\ulcorner A \vee (A \wedge B) \urcorner$, and so given (1) and (2), the propositions expressed by $\ulcorner A \urcorner$ and $\ulcorner A \vee (A \wedge B) \urcorner$ are identical, and so $A \equiv A \vee (A \wedge B)$.[2] Insofar as the theoretical role of propositional identity does not require more than sameness in modal profile of the interpreted sentences flanking the propositional identity sign, then such conclusions are easy to accept. However, it is natural to object that although $\ulcorner A \urcorner$ and $\ulcorner A \vee (A \wedge B) \urcorner$ are true in the same possible worlds, these sentences need not have the same subject-matter, where the subject-matter of an interpreted sentence is what that interpreted sentence is about. For instance, on its intended interpretation, the sentence 'It is raining, or both raining and snowing' is partly about it snowing, where the same cannot be said of the sentence 'It is raining'. Without giving up on a truth-conditional account of interpreted sentences as in (1), we may seek to refine our conception of a sentence's truth-condition, replacing (2) with some alternative, guided by the aim to accommodate sameness in

[1] Or one could take truth-conditions to be characteristic functions from worlds to truth-values.

[2] I will mostly use upper-case Roman letters to express propositions, relying on context to resolve use-mention ambiguities, while occasionally employing corner quotes for clarity.

subject-matter of the sentences flanking a propositional identity sign.

Setting aside whether primitivism is the right view of propositional identity or not, I will take the propositional identity operator '$\equiv$' to be a primitive term for the purposes of this paper. Instead of offering a definition, the present ambition will be to present a logic of propositional identity which is both motivated and constrained by the ambition to respect sameness of subject-matter in addition to necessary equivalence. By distinguishing necessarily equivalent propositions which differ in subject-matter, the theory of propositions developed below will provide novel theoretical resources. Rather than exploring any particular application of these resources, I will be concerned to trace the contours of the resulting hyperintensional theory of propositions by providing a definition of logical consequence as well as a derivability relation for the first-degree fragment of a propositional language.

It is worth contrasting an opposing strategy in which an intensional theory of propositions is merely augmented with a theory of subject-matter. For instance, assuming propositions to be sets of possible worlds, Lewis (1988a) identifies subject-matters with partitions of the set of all worlds, writing:

> A proposition is *about* a subject matter, and it is a subject matter *of* the proposition, iff the truth value of that proposition supervenes on that subject matter. [...] When we think of subject matters as partitions, we can say that P is about M iff each cell of M either implies or contradicts P. (p. 163)

The cells of a partition can be thought of as the different ways for the proposition in question to be true or false, where M *includes* N just in case every cell of N is a union of cells of M.[3] For instance, consider the proposition P_1 that there are more than a hundred stars. One such subject-matter of P_1 is the partition M_1 which consists of P_1 and its complement within the set of all worlds W, whereas another partition M_2 groups worlds together into cells which have the same number of stars. Accordingly, M_2 may be said to include M_1. Not only is there no unique subject-matter which a given proposition is about, Lewis (1988a, p. 171-2) admits that non-contingent propositions are about every subject-matter, "since there is no way at all for two worlds to give it different truth values, *a fortiori* there is no way for two worlds to give it different truth values without differing with respect to the subject matter."[4] Thus 'The gold atom α has 79 protons', '2+2=5', and all instances of $\ulcorner A \vee \neg A \urcorner$ and $\ulcorner B \wedge \neg B \urcorner$ express propositions which, according to Lewis, are about all subject-matters. Additionally, by identifying necessarily equivalent propositions, Lewis makes all necessarily equivalent propositions about the same subject-matters. For instance, $A \vee (A \wedge B)$, $A \wedge (A \vee B)$, and A will have the same subject-matters for any A and B, and so Lewis cannot accommodate the apparent difference in

[3] Lewis (1988b) speaks of inclusion, but one could read $\ulcorner M$ includes $N\urcorner$ as $\ulcorner M$ *refines* $N\urcorner$.

[4] Lewis (1988a, p. 164) does introduce the concept of *least subject-matters*, but admits that there need not always be a least subject-matter for a given proposition.

subject-matter between the proposition that it is raining and the proposition that it is raining or both raining and snowing.

Abstracting from the details of Lewis' account, the broader strategy aims to capture differences in subject-matter while assuming an intensional theory of propositions, asking for which values of X does the product $\mathcal{P}(W) \times X$ draw enough distinctions in order to encode differences in subject-matter.[5] As fruitful as intensional theories of propositions have been for many different applications, no such account can accommodate differences in subject-matter between necessarily equivalent propositions. As Perry (1989) writes:

> [T]he problem of necessary equivalent propositions is simply a fly bottle that did not have to be flown into. The solution is to fly out, not to argue that, all things considered, maybe it is not such a bad bottle to be in. (p. 191)

Rather than attempting to present an intensional theory of propositions, or else to augment an intensional theory of propositions with a theory of subject-matters, I will defend a hyperintensional theory of propositions which does not presume that the particular form of hyperintensionality in question can or should be factored into intensional and non-intensional components.[6] Before attempting to provide such an account, it will be important to motivate criteria for an adequate theory of propositional identity. Accordingly, §2 will present a number of principles which I will assume that an adequate account of subject-matter ought to include along with a minimal theory of propositional identity, where the adoption of these principles will guide our departure from standard Boolean theories of propositional identity. By drawing on the resources of Kit Fine's state semantics, §3 will present a hyperintensional theory of propositions which accommodates differences in subject-matter, comparing the result with Fine's (2016, 2017a,b) theory of regular propositions in §4.

After presenting a first-degree logic for propositional identity (PI^1) in §5, I will contrast Correia's (2016) logic of generalised identity (GI) which I show has the unwanted consequence of making negation an opaque operator. I will conclude in §6 by extending PI^1 to include axioms and rules of inference for a subject-matter operator, providing not only a broader theory of subject-matter, but one in which we may derive the first degree fragment of Angell's (1989) logic of analytic containment (AC) in addition to comparing Fine's (2016, 2017b,c, 2020) account of subject-matter. In §7, I will provide a few formal results which will be of use at various points throughout the paper.

[5] See Yablo (2014), Hawke (2018), Berto (2019) for other theories of this kind.

[6] As Perry (1989, p. 176) also observes, "the view that language was basically intensional, is older than possible-worlds semantics. Basically, intensions are entities that provide some principle of classification, and that have an identity, independently of the objects so classified." Rather than attempting to uproot the already entrenched practice of taking 'intensions' to be functions from possible-worlds to truth-values, I will maintain the spirit of Perry's critique of a possible worlds understanding of intensions by developing a hyperintensional alternative.

2 Subject-Matter

In order to facilitate the presentation of a theory of subject-matter, it will help to introduce the sentential operator 'σ', where σA is the subject-matter of A. Whereas Fine (2016, 2017b,c, 2020) takes subject-matters to be states-of-affairs, or just *states*, I will adopt a propositional account of subject-matter, whereby the subject-matter of an interpreted sentence is a proposition.[7] For instance, the subject-matter of 'It is raining' will be a proposition to do with the rainy weather. More specifically, §6 will argue that there is good reason to read $\ulcorner\sigma A\urcorner$ informally as ⌜It is partially the case that A or partially not the case that A⌝. By then letting $A \Leftrightarrow B := \sigma A \equiv \sigma B$, where we may read $\ulcorner A \Leftrightarrow B\urcorner$ informally as ⌜A and B have the same subject-matter⌝, I will adopt the following:

S1 $\neg A \Leftrightarrow A$.

S2 $A \wedge B \Leftrightarrow A \vee B$.

S3 $A \wedge A \Leftrightarrow A$.

S4 $A \wedge B \Leftrightarrow B \wedge A$.

S5 $\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$.

S6 $\neg(A \vee B) \Leftrightarrow (\neg A \wedge \neg B)$.

S7 $A \wedge (B \wedge C) \Leftrightarrow (A \wedge B) \wedge C$.

Obj $(A \equiv B) \rightarrow (A \Leftrightarrow B)$.

Setting aside differences in formalisation, the principles above are widely accepted.[8] By **S1** and **S2**, the converse of **Obj** may fail since $\neg A \equiv A$ and $A \wedge B \equiv A \vee B$ need not hold. Rather, **Obj** makes sameness of subject-matter a necessary but insufficient condition for propositional identity. Nevertheless, **Obj** asserts that what an interpreted sentence is about is solely a function of the proposition expressed by that sentence on its interpretation irrespective of the features unique to that sentence, or the concepts by which that proposition is expressed. For instance, given that for Hesperus to be rising just is for Phosphorus to be rising, it follows by **Obj** that the subject matter of Hesperus is rising is the same as the subject-matter of Phosphorus is rising. In slogan, the present theory takes subject-matter to be independent of guise.

I will refer to theories of subject-matter which grant **Obj** as *objective*, and theories which reject **Obj** as *representational*. By contrast with objective theories, representational theories make subject-matter at least partially a function of the means by which a given proposition is expressed. For instance, suppose that one were to take 'Hesperus is rising' and 'Phosphorus is rising' to differ in subject-matter despite expressing the same proposition. Insofar as the subject-matter of an interpreted sentence is what that sentence is about, what 'Hesperus is rising' and 'Phosphorus is rising' are about must

[7] Although states will be taken to be objects, one may think of the states in the intended model as propositions, for states are objects which may either obtain or fail to obtain.

[8] Even in adopting an objectual account of subject matter where $\ulcorner\sigma A\urcorner$ is taken to be a singular term, the principles above may be maintained by instead defining $A \Leftrightarrow B := \sigma A = \sigma B$. See Lewis (1988a), Perry (1989), Yablo (2014), Hawke (2018), Fine (2020) for accounts which are committed to analogues of the principles above on appropriate understandings of '$\Leftrightarrow$' and 'σ'.

differ. But this strikes a false note, for both sentences appear to be entirely about the movement of the same astrological body relative to one's position. Since the sentences 'Hesperus is rising' and 'Phosphorus is rising' express the same proposition, the only remaining differences between these sentences are representational in nature, having to do with the means by which each sentence expresses the same proposition.[9] However, it would be inappropriate to identify the subject-matter of an interpreted sentence with the means by which that sentence expresses the proposition that it does, at least insofar as the subject-matter of an interpreted sentence is what that sentence is about. For instance, neither 'Hesperus is rising' nor 'Phosphorus is rising' are about anything the least bit representational, and so the ways in which these sentences differ cannot be identified with their different subject-matters. Nevertheless, a representational theory of subject-matter could take the subject-matter of an interpreted sentence to be a function of the means by which that sentence expresses a proposition. Instead of pursuing this line, I will develop an objective theory of subject-matter where the subject-matter of an interpreted sentence is a function of the proposition which that sentence expresses. Thus I will speak directly of the subject-matter of each proposition, whether or not there is a sentence which expresses that proposition on a given interpretation.

Even in granting **Obj**, this principle cannot provide positive determinations of which propositional identities hold on account of including the propositional identity sign in its antecedent. Nevertheless, it follows from **Obj** that any discrepancies in the subject-matters of A and B entail that A and B are distinct. In this way, a theory of subject-matter may inform our present aim to provide a theory of propositional identity. Additionally, we may draw the following connection between the subject-matter of A and the propositions which are wholly relevant to A, where $\ulcorner A \leq B \urcorner$ reads ⌜It being the case that A is wholly relevant to it being the case that B⌝, or for ease, $\ulcorner A$ is relevant to $B \urcorner$:

Rel $(A \Leftrightarrow B) \rightarrow [(C \leq A) \rightarrow (C \leq B)]$.

If A and B share the same subject-matter— if what A and B are about is the same— then whatever is relevant to A must also be relevant to B. Together, **Obj** and **Rel** provide a means by which to distinguish propositions, for given some C where $C \leq A$ but $C \nleq B$, it follows that $A \not\Leftrightarrow B$ by **Rel**, and so $A \not\equiv B$ follows by **Obj**. Of course, such evaluations will depend on judgements about relevance. Nevertheless, **Obj** and **Rel** provide an additional basis by which to evaluate propositional identity claims for truth.

Recall the claim from before that although A and $A \vee (A \wedge B)$ have the same modal profile, they may fail to share the same subject-matter. Even more starkly, $A \vee \neg A$ and $B \vee \neg B$ may diverge completely in subject-matter despite both being necessary. For instance, the sentences 'I am sitting or not sitting' and 'Grass is green or not green' both express necessary propositions despite

[9] In particular, one might consider the syntactic differences between sentences, or else the different concepts thereby expressed.

having entirely distinct subject-matters, where neither is even partially about the other. In order to account for the possible divergence in subject-matter between $A \vee \neg A$ and $B \vee \neg B$, we may observe that even without giving a full theory of relevance, it is natural to accept the following principle:

L1 $A \preceq A \vee B$ **L2** $A \preceq A \wedge B$

L3 $B \preceq A \vee B$ **L4** $B \preceq A \wedge B$

Given some A and B for which $A \npreceq B \vee \neg B$, we know by **L1** that $A \preceq A \vee \neg A$, and so $(A \vee \neg A) \not\equiv (B \vee \neg B)$ follows by **Obj** and **Rel**, where a similar argument concludes that $(A \wedge \neg A) \not\equiv (B \wedge \neg B)$ on the basis of **L2** in place of **L1** in the previous argument.[10] For instance, my sitting fails to be wholly relevant to grass being green or not green, though of course my sitting is relevant to me sitting or not sitting. Thus by **Obj** and **Rel**, it is not the case that for me to be sitting or not sitting just is for grass to be green or not green.

Similar arguments may be given against adopting the absorption laws. Given some A and B where $B \npreceq A$ but $B \preceq [A \vee (A \wedge B)]$, it follows by **Obj** and **Rel** that $A \not\equiv A \vee (A \wedge B)$.[11] For instance, although it snowing is relevant to it raining or both raining and snowing, it snowing fails to be relevant to it raining, and so by **Obj** and **Rel**, it is not the case that for it to be raining just is for it to be raining or both raining and snowing. A similar argument shows that not all instances of $A \equiv A \wedge (A \vee B)$ hold. Given these considerations, we find reason to take exception to the following Boolean identities:

#Necs $(A \vee \neg A) \equiv (B \vee \neg B)$. **#Abs1** $A \equiv A \vee (A \wedge B)$.

#Imps $(A \wedge \neg A) \equiv (B \wedge \neg B)$. **#Abs2** $A \equiv A \wedge (A \vee B)$.

Whereas intensional theories of propositions are *Boolean* insofar as they affirm all of the Boolean identities, I will develop a non-Boolean alternative. In accordance with a conservative methodology, I will maintain as many of the Boolean identities as possible without overlooking differences in subject-matter. More specifically, I will assume that just as differences in either the subject-matter or the modal profile of A and B provide a reason to distinguish A and B, sameness in both the subject-matter and the modal profile of A and B provides at least a defeasible reason to maintain their identity.

In order to begin to evaluate the broader space of propositional identities, I will assume that anything deserving of the title 'propositional identity' ought to satisfy all instances of the following principles:

[10] Given the definition of relevance presented in §6, we may show that $A \preceq B \vee \neg B$ just in case $A \preceq B$, where of course there are some A and B where $A \npreceq B$, and so $A \npreceq B \vee \neg B$.
[11] One may show that $B \preceq [A \vee (A \wedge B)]$ as well as its dual is a theorem of PI^1_σ given in §6.

Ref $A \equiv A.$ **Trans** $(A \equiv B) \rightarrow [(B \equiv C) \rightarrow (A \equiv C)].$

Sym $(A \equiv B) \rightarrow (B \equiv A).$ **Imp** $(A \equiv B) \rightarrow (A \rightarrow B).$

Rejecting any of the principles above would be to change the topic from propositional identity to something else entirely. Additionally, given the present concern with worldly propositions, I will restrict attention to propositional operators which are insensitive to the means by which propositions are expressed. More precisely, we may say that an operator $\mathcal{Q}$ is *transparent* in a language $\mathcal{L}$ just in case all instances of the following principle hold, where $\ulcorner \vec{O}_{(B/A)} \urcorner$ is the result of freely substituting $\ulcorner B \urcorner$ for $\ulcorner A \urcorner$ in the sequence of $\mathcal{Q}$'s operands $\ulcorner \vec{O} \urcorner$:

Func $(A \equiv B) \rightarrow [\mathcal{Q}(\vec{O}) \equiv \mathcal{Q}(\vec{O}_{(B/A)})].$

Intuitively, a sentential operator is transparent in a language just in case it expresses a propositional function, where the output is determined solely by the inputs independent of the means of expressing those inputs. A language $\mathcal{L}$ is *transparent* just in case every operator $\mathcal{Q}$ in $\mathcal{L}$ is transparent in $\mathcal{L}$.

Given the present concern with the structure of the space of propositions independent of the structure of the different means of representing those propositions, I will restrict consideration to transparent languages throughout. As **P2** in §7 shows, any propositional language $\mathcal{L}$ which includes the extensional connectives '$\neg$', '$\wedge$', and '$\vee$' along with the propositional identity operator '$\equiv$' is transparent just in case the following principle holds without exception in $\mathcal{L}$, where $\ulcorner C_{(B/A)} \urcorner$ is the result of freely substituting $\ulcorner B \urcorner$ for $\ulcorner A \urcorner$ in $\ulcorner C \urcorner$:

LL $(A \equiv B) \rightarrow (C \rightarrow C_{(B/A)}).$

The principle above expresses Leibniz's law of the indiscernibility of identicals whereby identicals satisfy the same conditions. Instead of adopting **LL** as an independent assumption, **P1** and **P2** in §7 show that so long as **Ref** and **Imps** hold without exception, **Sym**, **Trans**, and **LL** follow from the stipulation that the language under consideration is transparent. Since the restriction to transparent languages was motivated by the concern to study the structure of the space of worldly propositions independent of any representational difference in thought or language, **LL** may be taken to inherit the same motivation given the ambition to provide a theory of identity for worldly propositions.

It is important to stress that in articulating a theory of propositional identity, we need only take a stand on the transparency of a limited range of operators. In particular, I will take the operators for conjunction, disjunction, negation, and subject-matter to be transparent. Letting an operator be *opaque* in $\mathcal{L}$ just in case it is not transparent in $\mathcal{L}$, the present paper need not take a stand on whether there are genuine cases of opacity, though I take it that there are such genuine cases, where it is in virtue of this fact that synonymy is much more fine-grained than propositional identity.[12]

[12] See Dorr (2016), Bacon (2019), Bacon (2019), and Caie et al. (2019) for recent discussion of opacity, as well as §6 for further comparison between synonymy and propositional identity.

Given that all instances of **LL** hold without exception in a language with operators for conjunction, disjunction, negation, and subject-matter, we may derive **S3** – **S7** from the following identities by means of classical reasoning:

A1 $A \wedge A \equiv A$.

A2 $A \vee A \equiv A$.

A3 $A \wedge B \equiv B \wedge A$.

A4 $A \vee B \equiv B \vee A$.

A5 $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$.

A6 $(A \vee B) \vee C \equiv A \vee (B \vee C)$.

A7 $\neg(A \wedge B) \equiv (\neg A \vee \neg B)$.

A8 $\neg(A \vee B) \equiv (\neg A \wedge \neg B)$.

A9 $A \equiv \neg\neg A$.

Moreover, given **S1** – **S7**, we may show that each of the identities above respects sameness of subject-matter. Given that these identities also respect necessary equivalence, a conservative methodology recommends adopting these principles in the absence of countervailing considerations. Suppose that one were to attempt to support the following principles by an analogous argument:

#Dist1 $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$.

#Dist2 $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$.

We may observe that the principles for subject-matter given so far do not provide any means by which to evaluate whether the identities above respect sameness of subject-matter. Since intuitive judgements about which principles for subject-matter hold without exception can only carry us so far, the following section will present a semantic theory for a propositional language, providing a systematic means of surveying the total space of possible counterexamples to propositional identity claims. As I will show in §4, there are strong abductive reasons to exclude **#Dist1** and **#Dist2** from the logic of propositional identity on account of admitting a compelling class of counterexamples.

3 State Semantics

In giving up the Boolean identities **#Necs**, **#Imps**, **#Abs1**, and **#Abs2**, it remains to provide an alternative non-Boolean theory of propositions. Instead of attempting to axiomatise propositional identity by means of intuition alone, this section will draw on Kit Fine's state semantics in order to provide a theory of propositions which is sensitive to hyperintensional differences in subject-matter while satisfying the principles adopted above.

For simplicity, I will focus on the first-degree fragment of the propositional language $\mathcal{L} = \langle \mathbb{L}, \neg, \vee, \wedge, \equiv \rangle$ with sentence letters $\mathbb{L} = \{p_i : i \in \mathbb{N}\}$, postponing consideration of extensions which include a subject-matter operator to §6. Given $\mathcal{L}$, we may define the *extensional sentences* of $\mathcal{L}$ as follows, where $p \in \mathbb{L}$:

$$A ::= p \mid \neg A \mid A \wedge A \mid A \vee A.$$

Letting $\texttt{ext}(\mathbb{L})$ be the set of extensional sentences in $\mathcal{L}$, I will take $A \equiv B$ to be an *identity sentence* in $\mathcal{L}$ for any $A, B \in \texttt{ext}(\mathbb{L})$, where $\texttt{id}(\mathbb{L})$ is the set of identity sentences in $\mathcal{L}$. Following Fine (2017a,b,c), we may take a *state space* to be an ordered pair $\mathcal{S} = \langle S, \sqsubseteq \rangle$, where S is a non-empty set, $\sqsubseteq$ is a partial order on S, and $\mathcal{S}$ forms a complete lattice which is defined in the usual way:

Upper Bound: $s \in S$ is an *upper bound* of $X \subseteq S$ *iff* $x \sqsubseteq s$ for every $x \in X$.

Least Upper Bound: s is a *least upper bound* of $X \subseteq S$ *iff* s is an upper bound of X, and $s \sqsubseteq y$ for every upper bound y of X.

Complete Lattice: $\mathcal{S}$ is a *complete lattice iff* every $X \subseteq S$ has a least upper bound.

As the uniqueness of a least upper bound for any set X is readily established, we may refer to the least upper bound $\bigsqcup X$ of X as the *fusion* of the states in X, designating the *full state* by $\bigsqcup S = \blacksquare$ and the *null state* by $\bigsqcup \varnothing = \square$. It remains to employ these resources to interpret the identity sentences in $\texttt{id}(\mathbb{L})$ in a manner which accords with the principles defended above.

Given a state space $\mathcal{S} = \langle S, \sqsubseteq \rangle$, I will follow Fine (2017a, p. 629) in taking propositions to be ordered pairs of sets of states which satisfy some number of constraints depending on the application. Whereas the following section will raise general problems for the theory of regular propositions which Fine (2016, 2017a) develops, the present section will present a competing alternative which I will argue is well suited to the application at hand. In particular, consider the space of normal propositions which may be defined over any state space $\mathcal{S}$:

Normal Contents: $\mathbb{C}_{\mathcal{S}} = \{X \subseteq S : \bigsqcup Y \in X \text{ for all nonempty } Y \subseteq X\}$.

Normal Propositions: $\mathbb{P}_{\mathcal{S}} = \{\langle X, Y \rangle : X, Y \in \mathbb{C}_{\mathcal{S}}\}$.[13]

Given any state space $\mathcal{S} = \langle S, \sqsubseteq \rangle$, an $\mathcal{S}$*-model* of $\mathcal{L}$ is any ordered triple $\mathcal{M} = \langle S, \sqsubseteq, |\cdot| \rangle$ where $|p| = \langle |p|^+, |p|^- \rangle$ with $|p|^\pm \subseteq S$ for every $p \in \mathbb{L}$.[14] An $\mathcal{S}$-model $\mathcal{M}$ is *normal* just in case $|p| \in \mathbb{P}_{\mathcal{S}}$ for every $p \in \mathbb{L}$. We may then take $\mathcal{N}$ to be the class of all normal $\mathcal{S}$-models of $\mathcal{L}$ for any state space $\mathcal{S}$, employing set notation where it is convenient.[15] In order to identify which sentences are true in all normal models of $\mathcal{L}$, we must first provide appropriate semantic clauses for the primitive operators included in the language.

Following Fine (2016, 2017c), we may recursively define exact verification $\Vdash$ and exact falsification $\dashV$ by means of the *inclusive semantics* given below, letting $t.d := \bigsqcup\{t, d\}$ for ease of exposition:

[13] I will discuss Correia's (2016) related view in §5 which Fine (2017a, p. 629) also cites.

[14] One could add top and bottom elements to the language where $\mathcal{S}$-models are required to assign $|\top| = \langle S, \varnothing \rangle$ and $|\bot| = \langle \varnothing, \{\square\} \rangle$ which are the top and bottom elements with respect to $\leqslant$, letting $\overline{\top} := \neg\top$ and $\underline{\bot} := \neg\bot$ be the top and bottom elements with respect to $\sqsubseteq$. By contrast, Fine (2017a, p. 630) takes $\langle S, \{\blacksquare\} \rangle$ to be a top element on account of excluding consideration of all vacuous propositions. See §4 below for further discussion.

[15] Though nothing turns on this point, I will officially assume a no-class theory of classes.

$(p)^+$ $\mathcal{M}, s \Vdash p$ *iff* $s \in |p|^+$. $(\neg)^+$ $\mathcal{M}, s \Vdash \neg A$ *iff* $\mathcal{M}, s \dashv\Vert A$.

$(p)^-$ $\mathcal{M}, s \dashv\Vert p$ *iff* $s \in |p|^-$. $(\neg)^-$ $\mathcal{M}, s \dashv\Vert \neg A$ *iff* $\mathcal{M}, s \Vdash A$.

$(\wedge)^+$ $\mathcal{M}, s \Vdash A \wedge B$ *iff* $s = t.d$ where $\mathcal{M}, t \Vdash A$ and $\mathcal{M}, d \Vdash B$.

$(\wedge)^-$ $\mathcal{M}, s \dashv\Vert A \wedge B$ *iff* $\mathcal{M}, s \dashv\Vert A$ or $\mathcal{M}, s \dashv\Vert B$ or $\mathcal{M}, s \dashv\Vert A \vee B$.[16]

$(\vee)^+$ $\mathcal{M}, s \Vdash A \vee B$ *iff* $\mathcal{M}, s \Vdash A$ or $\mathcal{M}, s \Vdash B$ or $\mathcal{M}, s \Vdash A \wedge B$.

$(\vee)^-$ $\mathcal{M}, s \dashv\Vert A \vee B$ *iff* $s = t.d$ where $\mathcal{M}, t \dashv\Vert A$ and $\mathcal{M}, d \dashv\Vert B$.

As a useful heuristic, we may consider an intended state space $\mathcal{S}_I$, where S_I is the set of all *states-of-affairs*, and $\sqsubseteq_I$ is a parthood relation. Instead of considering which sentences are true or false in which possible worlds, where worlds contain a great number of things which are completely irrelevant to any given sentence, exact verification and falsification is a matter of which sentences are "made" true or false by which states, where the truth-makers and falsity-makers must be wholly relevant to the sentences that they make true or false, respectively.[17] Although the exact verifiers for a conjunction are determined by the exact verifiers for its conjuncts, where similarly the exact verifiers for a disjunction are determined by the exact verifiers for its disjuncts, the same cannot be said for negation.[18] In particular, one cannot take any state which does not exactly verify a sentence to exactly verify its negation, at least insofar as states are required to be wholly relevant to the sentences which they exactly verify or falsify. It is for this reason that the state semantics assumes a bilateral form, extending consideration to exact falsifiers in addition to exact verifiers so as to identify the exact verifiers (falsifiers) for a sentence with the exact falsifiers (verifiers) for the negation of that sentence. In this respect, the inclusive semantics makes an important addition to (1) given in §1 by including falsity-conditions alongside truth-conditions.

Constructing the state semantics around the idea that states are to be wholly relevant to the sentences which they exactly verify or falsify also makes the notions of exact verification and falsification non-monotonic. Focusing on exact verification, Fine (2017a) brings this point out as follows:

> For it is to be a general requirement on verification that a verifier should be relevant as a whole to the statement that it verifies; and in extending a verifier with additional material, this holistic relevance of the verifier to the statement may be lost. (p. 626)

For instance, if the state t of Julieta thinking exactly verifies the sentence 'Julieta is thinking', and the state d of Julieta writing exactly verifies the sentence 'Julieta is writing', then the fusion $t.d$ fails to be wholly relevant to 'Julieta is thinking' as well as to 'Julieta is writing' on account of including

[16] Removing the final disjunct from $(\wedge)^-$ and $(\vee)^+$ yields the non-inclusive semantics.

[17] See Fine (2017a,b,c) for related discussion, as well as §4 below.

[18] Instead of considering what is relevant to the truth (falsity) of a sentence, one may consider what is relevant to either the truth or falsity of a sentence. See §6 for related discussion.

something irrelevant in each case. Nevertheless, the fusion state $t.d$ exactly verifies the conjunction 'Julieta is thinking and writing' on account of being a fusion of exact verifiers for each of its conjuncts, where similarly, any fusion of exact falsifiers for 'Julieta is thinking' and 'Julieta is writing' will exactly falsify the disjunction 'Julieta is thinking or writing'.

It remains to consider the exact verification clause for disjunction, and the exact falsification clause for conjunction. We may begin by observing that a disjunction is exactly verified by the exact verifiers for either of its disjuncts, and similarly, a conjunction is exactly falsified by the exact falsifiers for either of its conjuncts. Additionally, the inclusive semantics respects the claims that: (T) any exact verifier for a conjunction $A \wedge B$ will also be an exact verifier for the disjunction $A \vee B$; and, (F) any exact falsifier for $A \vee B$ will also be an exact falsifier for $A \wedge B$. In order to justify these latter additions, consider:

Uniformity: A class $\mathcal{K}$ of models of $\mathcal{L}$ is *uniform iff* for any $\mathcal{S}$-model $\mathcal{M} \in \mathcal{K}$ and $A \in \texttt{ext}(\mathbb{L})$, there is an $\mathcal{S}$-model $\mathcal{M}_\star \in \mathcal{K}$ and $p \in \mathbb{L}$ where $|p|^+_\star = \{s \in \mathcal{M} : \mathcal{M}, s \Vdash A\}$ and $|p|^-_\star = \{s \in \mathcal{M} : \mathcal{M}, s \dashV A\}$.

It is natural to require the class of models over which $\mathcal{L}$ is to be interpreted to be uniform since nothing about the sentence letters in $\mathbb{L}$ should prevent them from expressing the same propositions expressed by the complex sentences in $\texttt{ext}(\mathcal{L})$. Without requiring uniformity to hold, the law of uniform substitution is liable to fail, where uniform substitution is a natural desideratum for any logic. However, were one to give up (T), then the exact verifiers for a disjunction might fail to be closed under fusion, where the same may be said of the exact falsifiers for a conjunction were one to give up (F).[19] Given that the exact verifiers and falsifiers for any sentence letter are required to be closed under fusion, rejecting either (T) or (F) leads in each case to the non-uniformity of $\mathcal{N}$, thereby providing a reason to maintain the inclusive semantics.

Insofar as we are to restrict consideration to the class of normal models $\mathcal{N}$ of $\mathcal{L}$, uniformity provides a powerful reason to maintain both (T) and (F) in the inclusive semantics. In order to motivate the initial restriction to $\mathcal{N}$, it will help to set $|A|^+ = \{s \in S : \mathcal{M}, s \Vdash A\}$ and $|A|^- = \{s \in S : \mathcal{M}, s \dashV A\}$, adopting $|A| = \langle |A|^+, |A|^- \rangle$ as standard notation for the proposition that A expresses in $\mathcal{M}$. We may then provide the following semantic clause for the first-degree identity sentences in $\texttt{id}(\mathbb{L})$, along with the definitions of logical consequence and validity for an arbitrary class of models $\mathcal{K}$:

($\equiv$) $\mathcal{M} \vDash A \equiv B$ *iff* $|A| = |B|$.

Logical Consequence: $\varphi \in \texttt{id}(\mathbb{L})$ is a $\mathcal{K}$-*logical consequence* of $\Gamma \subseteq \texttt{id}(\mathbb{L})$, i.e. $\Gamma \vDash_{\mathcal{K}} \varphi$, just in case for any $\mathcal{M} \in \mathcal{K}$, if $\mathcal{M} \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \vDash \varphi$.

Validity: φ is $\mathcal{K}$-*valid* just in case $\vDash_{\mathcal{K}} \varphi$.

[19] Fine (2016, p. 206) establishes this result in **Lemma 6**.

Suppose that instead of adopting the inclusive semantics, one were to give up (T) and (F), maintaining uniformity by also giving up the closure condition on the sets of states which make up propositions. As Fine (2017c, p. 563) observes, A and $A \wedge A$ may then diverge in their exact verifiers, where A and $A \vee A$ may diverge in their exact falsifiers, thereby producing counterexamples to **A1** and **A2**. However, since A agrees in both subject-matter and modal profile with $A \wedge A$ and $A \vee A$, we find no reason to distinguish between A and either $A \wedge A$ or $A \vee A$ given our present aims. Thus if the semantics is to validate **A1** and **A2**, then $A \wedge A$ and $A \vee A$ must have the same exact verifiers and falsifiers as A. It follows that the sets of exact verifiers and falsifiers for A must both be closed under finite fusion in $\mathcal{S}$, where a set of states X is *closed under finite fusion* in $\mathcal{S}$ just in case $t.d \in X$ whenever both $t, d \in X$.

Let a *finite fusion* model $\mathcal{M}$ of $\mathcal{L}$ be any $\mathcal{S}$-model of $\mathcal{L}$ where both $|p|^{\pm}$ are closed under finite fusion in $\mathcal{S}$. Insofar as there may be more than one exact verifier or falsifier for a sentence in a finite fusion model of $\mathcal{L}$, it follows that the exact verifiers and falsifiers for a sentence may contain more than what is strictly required to make that sentence true or false respectively, and so need not be minimal.[20] For instance, any fusion of two or more exact verifiers for A is also an exact verifier for A, and so will contain proper parts which exactly verifies A, where something similar may be said for a fusion of two or more exact falsifiers for A. Moreover, it is not clear what would motivate a restriction on overdetermination of this kind to merely finite fusions. For instance, given any real number $625 \leqslant x \leqslant 740$, we may take s_x to be the state of α reflecting light with wavelength x nanometers, where $R = \{s_x : 625 \leqslant x \leqslant 740\}$. Insofar as each $s_x \in R$ is an exact verifier for the sentence 'α reflects red light', we may admit that $\bigsqcup Y$ exactly verifies 'α reflects red light' for any nonempty $Y \subseteq R$.[21] In order for infinite fusions of exact verifiers (falsifiers) to be admitted as exact verifiers (falsifiers) for sentences, I have taken normal propositions to include sets of states closed under infinite rather than finite fusion, restricting attention to the class of models $\mathcal{N}$ in which normal propositions are assigned to sentence letters. We may then show by an simple induction proof that closure under infinite fusion extends to all extensional sentences in $\texttt{ext}(\mathbb{L})$.

Although I will not provide much in the way of an exploration of the modalised state spaces introduced by Fine (2017c), it is nevertheless worth considering a primitive distinction between *possible* and *impossible* states, where every part of a possible state is also required to be possible. We may then say that two states t and d are *compatible* just in case their fusion $t.d$

[20] Compare minimal situations as used by Kratzer (1989) and Heim (1990) among others, as well as exemplifiers which Kratzer (1998, 2002) later employs.

[21] Considerations of gunk also motivate taking fusions of sets of exact verifiers of any non-zero cardinality to be exact verifiers. If a gunky ice cube is entirely pink, then the state of any part of it being pink will exactly verify the sentence 'Part of the ice cube is pink', where if a is a part of b, then the state of a being pink is also part of the state of b being pink. It follows that for any non-null part of the ice cube a, the state of a being pink will consist of an infinite fusion of exact verifiers for the sentence 'Part of the ice cube is pink'. See von Fintel (2002) for discussion of a related issue for Kratzer's situation semantics.

is possible, and *incompatible* otherwise. For instance, although the state of my sitting and the state of my standing are both possible, their fusion is impossible, making these states incompatible. Insofar as sentences are to admit of incompatible exact verifiers (falsifier), we may observe that far from an obscure artefact of the framework, impossible states do important work in drawing hyperintensional distinctions. In particular, it is natural to assume that no exact verifier v and falsifier f for a single sentence A could ever be compatible. However, it follows that every exact verifier for $A \wedge \neg A$ is a fusion of an exact verifier and falsifier for A, and so must therefore be impossible. Nevertheless, we may observe that whenever the exact verifiers and falsifiers for A and B do not share any parts in common, $A \wedge \neg A$ and $B \wedge \neg B$ are exactly verified and falsified by different impossible states, as are $A \vee \neg A$ and $B \vee \neg B$. Thus $A \wedge \neg A$ and $B \wedge \neg B$ may be distinct despite sharing the same modal profile, where the same may be said for $A \vee \neg A$ and $B \vee \neg B$, and so as desired, neither **#Imps** nor **#Necs** are valid over $\mathcal{N}$.

In addition to including counterexamples to **#Imps** and **#Necs**, neither **#Abs1** nor **#Abs2** are $\mathcal{N}$-valid. In particular, we may consider the model $\mathcal{M}_A = \langle S_A, \subseteq, |\cdot|_A \rangle$ where $S_A = \mathcal{P}(\{a, b, c, d\})$ with $|p_1|_A = \langle \{\{a\}\}, \{\{b\}\} \rangle$ and $|p_2|_A = \langle \{\{c\}\}, \{\{d\}\} \rangle$, for pairwise distinct a, b, c, and d. We may then derive:

$$\begin{aligned} |p_1 \vee (p_1 \wedge p_2)|_A &= \langle \{\{a\}, \{a, c\}\}, \{\{b\}, \{b, d\}\} \rangle \\ |p_1 \wedge (p_2 \vee p_2)|_A &= \langle \{\{a\}, \{a, c\}\}, \{\{b\}, \{b, d\}\} \rangle. \end{aligned}$$

Since $\{\{a\}\} \neq \{\{a\}, \{a, c\}\}$ and $\{\{b\}\} \neq \{\{b\}, \{b, d\}\}$, it follows from ($\equiv$) that both $\mathcal{M}_A \not\models p_1 \equiv p_1 \vee (p_1 \wedge p_2)$ and $\mathcal{M}_A \not\models p_1 \equiv p_1 \wedge (p_1 \vee p_2)$. Since $\mathcal{M}_A \in \mathcal{N}$, neither **#Abs1** nor **#Abs2** are $\mathcal{N}$-valid as claimed above. In order to add texture to the present counterexample, we may take p_1 to be 'It is raining' and p_2 to be 'It is windy', where $\{a\}$ is a rainy weather state and $\{c\}$ is a windy weather state. It follows by the inclusive semantics that although $\{a, c\}$ exactly verifies 'It is raining and windy', and so $\{a, c\}$ also exactly verifies 'It is raining or both raining and windy', the same cannot be said for 'It is raining', thereby indicating a discrepancy between the set of exact verifiers for 'It is raining' and 'It is raining or both raining and windy'. Similar considerations show that the sentences 'It is raining' and 'It is raining and either raining or windy' have different exact verifiers, and so do not express the same proposition.

By invalidating **#Necs**, **#Imps**, **#Abs1**, and **#Abs2**, the inclusive state semantics satisfies the initial aim set out in §2 to provide a theory of propositional identity which respects differences in subject-matter. Given that the definition of $\mathcal{N}$-logical consequence is perfectly general, it is straightforward to compute the $\mathcal{N}$-validity of any of the propositional identity sentences in $\mathcal{L}$. We may then ask whether $\mathcal{N}$-validity yields an extensionally adequate theory of propositional identity given the aim to track subject-matter, while preserving as many of the Boolean identities as possible. Instead of attempting to provide a determinate answer for all identity sentences in $\texttt{id}(\mathbb{L})$, the following section will begin by presenting considerations in favour of excluding **#Dist1** and **#Dist2** from the logic of propositional identity.

4 Distribution Laws

In addition to the counterexamples to **#Necs**, **#Imps**, **#Abs1**, and **#Abs2**, the semantics also admits counterexamples to **#Dist1** and **#Dist2**. Letting $\mathcal{M}_D = \langle S_D, \subseteq, |\cdot|_D \rangle$ with $S_D = \mathcal{P}(\{a, b, c, d, e, f\})$ where $|p_1|_D = \langle \{\{a\}\}, \{\{b\}\} \rangle$, $|p_2|_D = \langle \{\{c\}\}, \{\{d\}\} \rangle$, and $|p_3|_D = \langle \{\{e\}\}, \{\{f\}\} \rangle$, for pairwise distinct a, b, c, d, e, and f, we may derive the following identities:

$$
\begin{aligned}
|p_1 \vee (p_2 \wedge p_3)|_D &= \langle \underline{\{\{a\}, \{c, e\}, \{a, c, e\}\}}, \{\{b, d\}, \{b, f\}, \{b, d, f\}\} \rangle \\
|(p_1 \vee p_2) \wedge (p_1 \vee p_3)|_D &= \langle \underline{\{\{a\}, \{a, c\}, \{a, e\}, \{c, e\}, \{a, c, e\}\}}, \{\{b, d\}, \{b, f\}, \{b, d, f\}\} \rangle \\
|p_1 \wedge (p_2 \vee p_3)|_D &= \langle \{\{a, c\}, \{a, e\}, \{a, c, e\}\}, \underline{\{\{b\}, \{d, f\}, \{b, d, f\}\}} \rangle \\
|(p_1 \wedge p_2) \vee (p_1 \wedge p_3)|_D &= \langle \{\{a, c\}, \{a, e\}, \{a, c, e\}\}, \underline{\{\{b\}, \{b, d\}, \{b, f\}, \{d, f\}, \{b, d, f\}\}} \rangle.
\end{aligned}
$$

Given that the underlined sets of exact verifiers (falsifiers) are not identical, we may conclude by ($\equiv$) that $\mathcal{M}_D \nvDash A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ and $\mathcal{M}_D \nvDash A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$, and so neither **#Dist1** nor **#Dist2** are $\mathcal{N}$-valid.[22] However, the intuitive basis for claiming that **#Dist1** and **#Dist2** do not respect sameness of subject-matter is not nearly as obvious as it is for the absorption laws. Since **#Dist1** and **#Dist2** respect sameness in modal profile, I will take there to be a presumption in favour of accepting **#Dist1** and **#Dist2** on grounds of parsimony, drawing fewer distinctions in the absence of countervailing considerations. Nevertheless, there are powerful abductive reasons for taking counterexamples such as $\mathcal{M}_D$ seriously, excluding **#Dist1** and **#Dist2** from the logic of propositional identity rather than restricting the space of models so as to rule out such counterexamples.

We may begin by observing that, *modulo* simplifying assumptions, the counterexample above commands a degree of intuitive appeal. For instance, let $\{a\}$ be the state of the ball being black, $\{c\}$ be the state of the ball being round, and $\{e\}$ be the state of the ball being iron, where p_1 is 'The ball is coloured', p_2 is 'The ball is shaped', and p_3 is 'The ball is metallic'. It follows that the fusion state $\{a, c\}$ of the ball being black and round fails to exactly verify 'The ball is coloured or both shaped and metallic'. This conclusion follows from the observations that: (1) $\{a, c\}$ includes $\{c\}$ as a part, and so does not exactly verify 'The ball is coloured' since $\{c\}$ is irrelevant; (2) $\{a, c\}$ is not a fusion of exact verifiers for 'The ball is shaped' and 'The ball is metallic', and so does not exactly verify 'The ball is shaped and metallic'; and (3) $\{a, c\}$ is not a fusion of exact verifiers for 'The ball is coloured' and 'The ball is shaped and metallic', and so does not exactly verify 'The ball is coloured and both shaped and metallic'. Without changing the inclusive semantics, $\{a, c\}$ does not exactly verify 'The ball is coloured or both shaped and metallic'. Nevertheless, it is easy to see that $\{a, c\}$ exactly verifies 'The ball is coloured or

[22] Correia (2016, p. 111-2) presents an analogous counterexample to **#Dist1** articulated in terms of his supersentence semantics, but does not extend similar considerations to **#Dist2** as above, claiming instead that **#Dist2** is valid. See §5 for further discussion.

shaped' on account of exactly verifying 'The ball is coloured and shaped', and so $\{a, c\}$ exactly verifies 'The ball is both coloured or shaped, and coloured or metallic'. Given these considerations, we may conclude that $\mathcal{M}_D$ corresponds to an intuitively compelling counterexample to **Dist1**, where something similar may be shown for **Dist2**, making it unnatural to exclude these cases.

Insofar as the present notion of validity may claim to be reasonably natural, arbitrary models of $\mathcal{L}$ should not be excluded from consideration by fiat alone. Rather than attempting to hand pick the models of $\mathcal{L}$ which are to be considered in evaluating the validity of identity sentences, one may attempt to provide motivation for adopting a principled restriction on the class of models. In particular, Fine (2016, 2017a) considers the following restriction:

Regular Contents: $\mathbb{C}^{\text{R}}_{\mathcal{S}} = \{X \in \mathbb{C}_{\mathcal{S}} : y \in X$ whenever $x \sqsubseteq y \sqsubseteq z$ for some $x, z \in X\}$.

Regular Propositions: $\mathbb{P}^{\text{R}}_{\mathcal{S}} = \{\langle X, Y\rangle : X, Y \in \mathbb{C}^{\text{R}}_{\mathcal{S}}\}$.

Regularity: An $\mathcal{S}$-model $\mathcal{M}$ is *regular iff* $\mathcal{M} \in \mathcal{N}$ and $|p| \in \mathbb{P}^{\text{R}}_{\mathcal{S}}$ for all $p \in \mathbb{L}$.

In order to motivate restricting consideration to regular models, Fine (2017a) appeals to the simplicity of the space of regular propositions, writing:

> Regular propositions have an especially simple form. For each such proposition P (if non-empty) will have a maximal verifier p, the fusion of all its verifiers, which we identify with its subject-matter— the agglomeration of the facts, so to speak, from which its verifiers are drawn; and it will also have various low lying verifiers, with all other verifiers lying above them. The proposition itself will then consist of all the states that lie between the low lying verifiers and the maximal verifier. Regular propositions are therefore subject to a limited form of monotonicity; given that a state verifies a regular proposition then so does any extension of the state as long as it lies within the subject-matter of the proposition. (p. 628-9)

It is worth considering the manner in which the limited form of monotonicity to which Fine refers fails to hold in the counterexamples to **Dist1** and **Dist2**. In particular, with respect to **Dist1**, the outlier states in $|(p_1 \vee p_2) \wedge (p_1 \vee p_3)|^+_D$ all lie between states in $|p_1 \vee (p_2 \wedge p_3)|^+_D$, despite failing to be members of $|p_1 \vee (p_2 \wedge p_3)|^+_D$, and so $|p_1 \vee (p_2 \wedge p_3)|^+_D$ fails to be regular. Something similar may be said for **Dist2** since $|p_1 \wedge (p_2 \vee p_3)|^-_D$ also fails to be regular.

Despite the fact that neither $|p_1 \vee (p_2 \wedge p_3)|^+_D$ nor $|p_1 \wedge (p_2 \vee p_3)|^-_D$ are regular, we may nevertheless observe that $\mathcal{M}_D$ is a regular model, at least insofar as all other sentences letters are assigned to regular propositions. Even so, the complex sentences $p_1 \vee (p_2 \wedge p_3)$ and $p_1 \wedge (p_2 \vee p_3)$ do not express regular propositions, and so the class of regular models $\mathcal{R}$ fails to be uniform over the inclusive semantics. Rather than giving up regularity, one may maintain uniformity by adopting the following alternative semantic clauses:

$(\wedge)^{-}_{\text{S}}$ $\quad \mathcal{M}, s \dashv\!\!|\; A \wedge B$ *iff* $t \sqsubseteq s \sqsubseteq d$ for some t and d where $\mathcal{M}, d \dashv\!\!|\; A \vee B$ and either $\mathcal{M}, t \dashv\!\!|\; A$ or $\mathcal{M}, t \dashv\!\!|\; B$.

$(\vee)^{+}_{\text{S}}$ $\quad \mathcal{M}, s \Vdash A \vee B$ *iff* $t \sqsubseteq s \sqsubseteq d$ for some t and d where $\mathcal{M}, d \Vdash A \wedge B$ and either $\mathcal{M}, t \Vdash A$ or $\mathcal{M}, t \Vdash B$.

I will refer to the result of replacing $(\wedge)^{-}$ and $(\vee)^{+}$ in the inclusive semantics with the clauses above as the *super inclusive semantics*. We may then show that $\mathcal{R}$ is uniform over the super inclusive semantics so that for any $\mathcal{S}$-model $\mathcal{M} \in \mathcal{R}$ and $A \in \texttt{ext}(\mathbb{L})$, the proposition $|A| \in \mathbb{P}^{\text{R}}_{\mathcal{S}}$. Additionally, we may show that **#Dist1** and **#Dist2** are $\mathcal{R}$-valid over the super inclusive semantics. However, these validities have come at the cost of a significant increase in the complexity of the semantics for conjunction and disjunction. Even supposing that the space of regular propositions could be shown to be simpler than the space of normal propositions, we must nevertheless weigh this increase in simplicity against the increase in complexity of the super inclusive semantics.

Given the super inclusive semantics together with a model $\mathcal{M} \in \mathcal{R}$, we may refer to the elements of $|A|^{+}$ as the *liberal verifiers* for A in $\mathcal{M}$, and refer to the elements of $|A|^{-}$ as the *liberal falsifiers* for A in $\mathcal{M}$.[23] In addition to the increase in complexity, it is natural to object that the super inclusive semantics does not capture the most compelling falsity-conditions for conjunction, nor truth-conditions for disjunction. For instance, in the example above, both $\{a\}$ and $\{c, e\}$ liberally verify 'The ball is coloured or both shaped and metallic', but the same cannot be said for $\{c\}$ considered on its own. However, given that $\{c\}$ is part of $\{c, e\}$, it follows that $\{a, c\}$ liberally verifies 'The ball is coloured or both shaped and metallic'. By contrast, $\{a, c\}$ does not exactly verify 'The ball is coloured or both shaped and metallic' on account of including $\{c\}$ as a part which makes no contribution to the truth of the sentence when considered on its own, or even when fused with $\{a\}$. Whereas $\{a, c, e\}$ overdetermines the truth of 'The ball is coloured or both shaped and metallic' on account of being the fusion of more than one state which makes the sentence true, the same cannot be said for $\{a, c\}$ since it is only when fused with $\{e\}$ that $\{c\}$ may be said to contribute to making 'The ball is coloured or both shaped and metallic' true. Accordingly, without being fused to $\{e\}$, the state $\{c\}$ is irrelevant to the truth of 'The ball is coloured or both shaped and metallic', and so $\{c\}$ without $\{e\}$ disqualifies $\{a, c\}$ as an exact verifier for that sentence.

Even if one were to relax the notion of relevance holding between states and the sentences which they make true or false, the super inclusive semantics faces a further difficulty. In particular, one must restrict attention to the class $\mathcal{R}^{+}$ of nonvacuous regular models of $\mathcal{L}$ which may be defined as:

Nonvacuous: An $\mathcal{S}$-model $\mathcal{M}$ is *nonvacuous iff* both $|p|^{\pm} \neq \varnothing$ for all $p \in \mathbb{L}$.

[23] Although liberal verifiers (falsifiers) may be defined as the convex closure of the exact verifiers (falsifiers) for a sentence, exact verification (falsification) cannot be defined in terms of liberal verification (falsification) since different sets may have the same convex closure.

Without restricting consideration to just the nonvacuous regular models, we may let $\mathcal{M}_E = \langle S_E, \subseteq, |\cdot|_E\rangle$ with $S_E = \mathcal{P}(\{a,b,c\})$ and $|p_1|_E = \langle\varnothing, \{\{a\}\}\rangle$, $|p_2|_E = \langle\{\{a\}\}, \varnothing\rangle$, and $|p_3|_E = \langle\{\{b\}\}, \{\{c\}\}\rangle$ for distinct a, b, and c, where:

$$\begin{aligned}|p_1 \wedge p_3|_E &= \langle\varnothing, \{\{a\},\{c\},\{a,c\}\}\rangle;\\ |p_1 \vee p_3|_E &= \langle\varnothing, \{\{a,c\}\}\rangle;\\ |p_2 \vee p_3|_E &= \langle\{\{a\},\{b\},\{a,b\}\}, \varnothing\rangle;\\ |p_2 \wedge p_3|_E &= \langle\{\{a,b\}\}, \varnothing\rangle.\end{aligned}$$

Since p_1 has no liberal verifiers, there are no liberal verifiers for $p_1 \wedge p_3$, and so no state lies between a liberal verifier for $p_1 \wedge p_3$ and a liberal verifier for either p_1 or p_3. Even though $\{b\}$ liberally verifies p_3, it follows by the super inclusive semantics for disjunction that there are no liberal verifiers for $p_1 \vee p_3$, where a similar line of reasoning explains why there are no liberal falsifiers for $p_2 \wedge p_3$ despite the fact that $\{c\}$ liberally falsifies p_3. However, this is far from natural. By contrast, the inclusive semantics for disjunction maintains that any exact verifier for a disjunct ought to immediately qualify as an exact verifier for the disjunction to which it belongs, where similarly, the inclusive semantics for conjunction takes every exact falsifier for a conjunct to immediately qualify as an exact falsifier for the conjunction to which it belongs. Letting a proposition be *vacuous* just in case it either has no exact verifiers or no exact falsifiers, we may refer to the unnatural effects induced by the super inclusive semantics as *vacuous annihilation*. Although a proponent of the super inclusive semantics could prevent vacuous annihilation from occurring by restricting consideration to the class of nonvacuous regular models $\mathcal{R}^+$, there is nothing to motivate this restriction aside from the ambition to avoid vacuous annihilation.[24]

In addition to being *ad hoc*, the restriction to nonvacuous models trades on the implicit assumption that the sentence letters in $\mathbb{L}$ are naturally interpreted over the space of nonvacuous propositions. However, we may show in opposition to this assumption that the restriction to nonvacuous propositions prevents the space of propositions from assuming an otherwise much more natural form. To begin with, one may expect that any theory of propositions ought to be *bounded* insofar as for any set of propositions U, there is a proposition B which entails every proposition in U, as well as a proposition T which is entailed by every proposition in U. By letting U be the set of all propositions, we are guaranteed the existence of an *upper bound* which is entailed by all propositions, as well as a *lower bound* which entails all propositions. For instance, in an extensional theory of propositions, the upper and lower bounds on the space of propositions are the only propositions— namely, *truth* and *falsity*— where we may model these by $\{1\}$ and $\varnothing$, with subset inclusion for entailment. Similarly, intensional theories of propositions may be modelled by $\mathcal{P}(W)$ for nonempty W, where subset inclusion is entailment, and the set all worlds W and the set of no worlds $\varnothing$ are the upper and lower bounds. Not only are such extensional

[24] For instance, Fine (2017a, p. 649) considers restricting attention to nonvacuous propositions, providing a number of results which turn on this assumption.

and intensional theories of propositions bounded, this fact follows from their *completeness*, whereby any set U of propositions has a least upper bound as well as a greatest lower bound with respect to entailment, where these correspond to semantic analogues of infinite disjunction and conjunction for the propositions in U, respectively. Although completeness and boundedness could be given up, there is nothing to recommend these further departures from tradition given the present ambition to provide a theory of propositions which respects differences in subject-matter. Thus in accordance with a conservative methodology, I will take there to be a presumption in favour of maintaining the boundedness and completeness of the present theory of propositions.

By contrast with extensional and intensional theories of propositions which are ordered by a single entailment relation, the present aim to accommodate differences in subject-matter yields a theory of propositions which admits of two distinct orders. To see where these orders come from, consider the abbreviated proofs from Boolean logics that *conjunctive-parthood* $A \sqsubseteq B := A \wedge B \equiv B$ and *disjunctive-parthood* $A \leqslant B := A \vee B \equiv B$ are converse relations:

$$\begin{aligned} A \sqsubseteq B \quad &\Rightarrow \quad A \wedge B \equiv B && \text{(def)} \\ &\Rightarrow \quad A \vee (A \wedge B) \equiv A \vee B && (\mathbf{LL}) \\ &\Rightarrow \quad A \equiv A \vee B && (\mathbf{\#Abs1}) \\ &\Rightarrow \quad B \vee A \equiv A && (\mathbf{A4,\ LL}) \\ &\Rightarrow \quad B \leqslant A. && \text{(def)} \end{aligned}$$

$$\begin{aligned} A \leqslant B \quad &\Rightarrow \quad A \vee B \equiv B && \text{(def)} \\ &\Rightarrow \quad A \wedge (A \vee B) \equiv A \wedge B && (\mathbf{LL}) \\ &\Rightarrow \quad A \equiv A \wedge B && (\mathbf{\#Abs2}) \\ &\Rightarrow \quad B \wedge A \equiv A && (\mathbf{A3,\ LL}) \\ &\Rightarrow \quad B \sqsubseteq A. && \text{(def)} \end{aligned}$$

Giving up **#Abs1** and **#Abs2** blocks the derivations provided above. Rather, $A \sqsubseteq B$ may hold without $B \leqslant A$ holding, and *vice versa*. For instance, although $A \sqsubseteq A \wedge B$ is valid, $A \wedge B \leqslant A$ may fail to hold, since $(A \wedge B) \vee A$ need not be wholly relevant to A, and so $(A \wedge B) \vee A \not\equiv A$. Similarly, although $A \leqslant A \vee B$ is valid, $A \vee B \sqsubseteq A$ need not hold, since $(A \vee B) \wedge A$ may fail to be wholly relevant to A, and so $(A \vee B) \wedge A \not\equiv A$. Whereas conjunctive-parthood and disjunctive-parthood are two ways of specifying the same order in a Boolean theory, these relations may come apart in the present setting.[25] Nevertheless, the points given above with regards to boundedness and completeness may be reiterated for both orders: with respect to $\sqsubseteq$ and $\leqslant$, the space of propositions ought to be bounded both above and below, where this feature follows from the stronger requirement that both orders form complete lattices.

[25] As I argue elsewhere, '$\leqslant$' and '$\sqsubseteq$' provide natural regimentations of constitutive readings of 'necessary for' and 'sufficient for'. See also Fine (2015) and Correia and Skiles (2019).

Insofar as disjunctive-parthood and conjunctive-parthood are taken to be complete (and so bounded) lattices, we may show that the spaces of both normal propositions $\mathbb{P}_{\mathcal{S}}$ as well as regular propositions $\mathbb{P}^{\mathrm{R}}_{\mathcal{S}}$ form bilattices, providing natural hyperintensional analogues of the Boolean lattices familiar from extensional and intensional logics. In order to bring this out, we may define semantic correlates of conjunctive-parthood, disjunctive-parthood, and negation as follows, where $X = \langle X^+, X^- \rangle$ and $Y = \langle Y^+, Y^- \rangle$ are propositions:

Essence: $X \sqsubseteq Y$ *iff*: (1) for every $b \in Y^+$ there is some $a \in X^+$ where $a \sqsubseteq b$;
(2) $a.b \in Y^+$ whenever $a \in X^+$ and $b \in Y^+$; and
(3) $X^- \subseteq Y^-$.

Ground: $X \leqslant Y$ *iff*: (1) for every $b \in Y^-$ there is some $a \in X^-$ where $a \sqsubseteq b$;
(2) $a.b \in Y^-$ whenever $a \in X^-$ and $b \in Y^+$; and
(3) $X^+ \subseteq Y^+$.

Inversion: $\neg\langle X^+, X^- \rangle = \langle X^-, X^+ \rangle$.

Given the definitions above, we may show for any normal model $\mathcal{M} \in \mathcal{N}$ that: (1) $\mathcal{M} \vDash A \sqsubseteq B$ just in case $|A| \sqsubseteq |B|$; (2) $\mathcal{M} \vDash A \leqslant B$ just in case $|A| \leqslant |B|$; and (3) $|\neg A| = \neg|A|$. Since $\mathcal{R} \subseteq \mathcal{N}$, these results also apply to the class of regular models. We may then consider the following definition:

Bilattice: A structure $\mathcal{B} = \langle \mathbb{P}, \sqsubseteq, \leqslant, \neg \rangle$ is *bilattice iff* $\langle \mathbb{P}, \leqslant \rangle$ and $\langle \mathbb{P}, \sqsubseteq \rangle$ are complete lattices where $\mathbb{P}$ contains at least two elements and $\neg$ is a unary operator which satisfies the conditions: (1) $\neg\neg X = X$; (2) $X \leqslant Y = \neg X \sqsubseteq \neg Y$; and (3) $X \sqsubseteq Y = \neg X \leqslant \neg Y$.

The definition above was originally presented by Ginsberg (1988, 1990), and studied extensively by Fitting (1989a,b, 1990, 1991, 1994, 2002), among others. Whereas both $\mathcal{B}_{\mathcal{S}} = \langle \mathbb{P}_{\mathcal{S}}, \sqsubseteq, \leqslant, \neg \rangle$ and $\mathcal{B}^{\mathrm{R}}_{\mathcal{S}} = \langle \mathbb{P}^{\mathrm{R}}_{\mathcal{S}}, \sqsubseteq, \leqslant, \neg \rangle$ may be shown to be bilattices, we may nevertheless observe that the same cannot be said of the corresponding spaces of nonvacuous propositions, thereby falling short of what otherwise belongs to a natural class of structures.[26]

Even if one were to give up boundedness, and hence completeness— taking the space of propositions to form an *unbounded bilattice* in the sense studied by Bou and Rivieccio (2011)— it is natural to maintain that each order forms a lattice over the space of propositions so that for any two propositions there is guaranteed to be both a least upper bound as well as a greatest lower bound.[27] However, spaces of nonvacuous propositions do not form bilattices (unbounded

[26] This is not to claim that philosophers ought to be bound to what mathematicians have found to be most natural. Rather, I take it that without powerful motivation to do otherwise, a conservative methodology recommends beginning by thoroughly investigating appropriate applications of the most natural mathematical resources that have already been developed.
[27] An *unbounded bilattice* consists of two lattices with at least two elements together with a unary operator satisfying the conditions (1) – (3) in *Bilattice* above.

or otherwise) since for any two propositions X and Y where no state is a part of either the exact verifiers or falsifiers for both X and Y, there may fail to be a lower bound for X and Y with respect to either order, and so no greatest lower bound for X and Y. Without admitting vacuous propositions, the space may at most be said to consist of two join-semilattices with at least two elements and a unary operator which satisfies the conditions (1) – (3) in the definition of a bilattice, though there is little to suggest that such structures make up a reasonably natural class.[28] Fine (2017a) makes a related observation, writing:

> The domain of propositions has the structure of a lattice from a classical point of view and the structure (or something like the structure) of a bilattice from the present point of view. (p. 643, ft. 10)

Instead of following Fine (p. 649) in focusing on nonvacuous propositions, establishing a range of results which turn on nonvacuity, I will restrict attention in what follows to the normal and regular bilattices $\mathcal{B}_{\mathcal{S}}$ and $\mathcal{B}^{\mathrm{R}}_{\mathcal{S}}$.

Given the arguments above, a proponent of a regular theory of propositions cannot adopt either an inclusive or super inclusive semantics, at least insofar as the uniformity of $\mathcal{R}$ is to be maintained while avoiding nonvacuous collapse. In order to identify a suitable semantics for a regular theory of propositions, it will help to define the propositional operators which are expressed by '$\wedge$' and '$\vee$' when interpreted over the inclusive semantics, where $X, Y \in \mathbb{P}_{\mathcal{S}}$:

Content Fusion: $J \sqcap K = \{x.y : x \in J,\ y \in K\}$.

Conjunction: $X \wedge Y = \langle X^+ \sqcap Y^+,\ X^- \cup Y^- \cup (X^- \sqcap Y^-)\rangle$.

Disjunction: $X \vee Y = \langle X^+ \cup Y^+ \cup (X^+ \sqcap Y^+),\ X^- \sqcap Y^-\rangle$.

Given the inclusive semantics, we may show that for any normal model $\mathcal{M} \in \mathcal{N}$, both: (I) $|A \wedge B| = |A| \wedge |B|$; and (II) $|A \vee B| = |A| \vee |B|$. Moreover, we may show that $X \wedge Y$ and $X \vee Y$ are the least upper bounds of $X, Y \in \mathbb{P}_{\mathcal{S}}$ with respect to $\sqsubseteq$ and $\leqslant$, specifying clear theoretical roles for conjunction and disjunction to play within any bilattice of propositions $\mathcal{B}_{\mathcal{S}}$.[29]

Although $\mathbb{P}_{\mathcal{S}}$ is closed under the operators $\wedge$ and $\vee$, the same cannot be said of $\mathbb{P}^{\mathrm{R}}_{\mathcal{S}}$, where it is this fact together with (I) and (II) which explains why $\mathcal{R}$ fails to be uniform over the inclusive semantics. In order to maintain a regular theory of propositions, Fine (2017a, p. 632) considers the convex closure of the exact verifiers and falsifiers specified by the inclusive semantics, where the resulting semantic operations may be shown to be equivalent to the following:

[28] Fine (2017a, p. 642) takes conjunction to be the greatest lower bound with respect to *containment*, flipping the perspective on conjunctive-parthood which I will maintain. See §6 for Fine's definition of containment along with a comparison to *Essence*.

[29] These results may be taken to show that the semantic relations given in *Essence* and *Ground* are indeed the semantic correlates of conjunctive-part and disjunctive-part, respectively. One may also consider operators $\otimes$ and $\oplus$ for the greatest lower bounds with respect to $\sqsubseteq$ and $\leqslant$, referring to these as *common essence* and *common ground*, respectively.

Span: $[J, K] = \{y : x \sqsubseteq y \sqsubseteq z \text{ for some } x \in J \text{ and } z \in K\}$.

Convex Conjunction: $X \mathbin{\text{ⓐ}} Y = \langle X^+ \sqcap Y^+,\ [X^- \cup Y^-, \{\bigsqcup(X^- \cup Y^-)\}]\rangle$.

Convex Disjunction: $X \mathbin{\text{ⓥ}} Y = \langle [X^+ \cup Y^+, \{\bigsqcup(X^+ \cup Y^+)\}],\ X^- \sqcap Y^-\rangle$.

We may then show that $\mathbb{P}^{\mathrm{R}}_{\mathcal{S}}$ is closed under ⓐ and ⓥ, where indeed X ⓐ Y and X ⓥ Y are the least upper bounds with respect to $\sqsubseteq$ and $\leqslant$ respectively, so long as X and Y are restricted to $\mathbb{P}^{\mathrm{R}}_{\mathcal{S}}$. In place of the inclusive semantics, I will refer to the result of disjoining the super inclusive semantic clauses and the inclusive semantic clauses as the *extremely inclusive semantics*. Given any regular model $\mathcal{M} \in \mathcal{R}$, we may then show that $|A \wedge B| = |A|$ ⓐ $|B|$ and $|A \vee B| = |A|$ ⓥ $|B|$ hold with respect to the extremely inclusive semantics, making $\mathcal{R}$ uniform over the extremely inclusive semantics. Moreover, neither $\wedge$ nor $\vee$ result in vacuous annihilation as observed above for the super inclusive semantics, making the extremely inclusive semantics a superior alternative, at least insofar as a regular theory of propositions is to be maintained.

Recall Fine's claim from before that regular propositions have an especially simple form. As we have seen, the space of nonvacuous regular propositions does not satisfy the definition of a bilattice, and diverges from extensional and intensional theories of propositions in being unbounded and incomplete, where it is not clear what would motivate such departures given our present aims. At the same time, admitting vacuous propositions makes $\mathcal{R}$ fail to be uniform over the inclusive semantics, motivating the super inclusive semantics which is at least uniform over $\mathcal{R}$. However, the super inclusive semantics gives rise to vacuous annihilation, providing a reason to adopt the much more complicated and less natural extremely inclusive semantics which avoids both of these defects. Nevertheless, the bilattice of regular propositions $\mathcal{B}^{\mathrm{R}}_{\mathcal{S}}$ satisfies both:

Interlaced: A bilattice $\mathcal{B} = \langle \mathbb{P}, \sqsubseteq, \leqslant, \neg\rangle$ is *interlaced iff* $(X \star Z) \circ (Y \star Z)$ if $X \circ Y$ where $\star \in \{\wedge^{\sqsubseteq}, \wedge^{\leqslant}, \vee^{\sqsubseteq}, \vee^{\leqslant}\}$ and $\circ \in \{\leqslant, \sqsubseteq\}$, where $\wedge^{\circ}$ and $\vee^{\circ}$ are the least upper bounds with respect to $\sqsubseteq$ and $\leqslant$.

Distributive: A bilattice $\mathcal{B} = \langle \mathbb{P}, \sqsubseteq, \leqslant, \neg\rangle$ is *distributive iff* whenever $\star, * \in \{\wedge^{\sqsubseteq}, \wedge^{\leqslant}, \vee^{\sqsubseteq}, \vee^{\leqslant}\}$, then $X \star (Y * Z) = (X \star Y) * (X \star Z)$.

By contrast with $\mathcal{B}^{\mathrm{R}}_{\mathcal{S}}$, the bilattice of normal propositions $\mathcal{B}_{\mathcal{S}}$ may fail to be distributive, and so non-interlaced since— as Fitting (1990) observes— every distributive bilattice is interlaced. Although being distributive and interlaced are elegant properties for a bilattice to have, such virtues must be weighed against the increase in complexity of the extremely inclusive semantic clauses. Even more importantly, we must ask which properties are appropriate given the application at hand. In addition to their added complexity, I take the counterexamples discussed at the beginning of the present section to show that the extremely inclusive semantics fails to provide natural semantic clauses for conjunction and disjunction for the same reasons given for the super inclusive semantics. In particular, one must weaken the manner in which states are required to be relevant to the sentences that they verifier or falsifier, adopting appropriate liberalisations of verification and falsification in place of the exact analogues. It is on these grounds that I will continue to maintain the inclusive semantics, extending consideration to all models in $\mathcal{N}$ as required by uniformity, and so will exclude **#Dist1** and **#Dist2** from the logic of propositional identity. Nevertheless, *Interlaced* and *Distributive* articulate at least one sense in which regular propositions may be said to enjoy a degree of simplicity which normal propositions do not, thereby corroborating Fine's claims above.

Even in giving up **#Dist1** and **#Dist2** in addition to **#Abs1** and **#Abs2**, on account of admitting counterexamples when evaluated over $\mathcal{N}$ given the inclusive semantics, the following principles may be shown to be $\mathcal{N}$-valid:

A10 $A \wedge (A \vee B) \equiv A \vee (A \wedge B)$.

A11 $A \vee (B \wedge C) \leqslant (A \vee B) \wedge (A \vee C)$.

A12 $A \vee (B \wedge C) \sqsubseteq (A \vee B) \wedge (A \vee C)$.

A13 $A \wedge (B \vee C) \leqslant (A \wedge B) \vee (A \wedge C)$.

A14 $A \wedge (B \vee C) \sqsubseteq (A \wedge B) \vee (A \wedge C)$.

Whereas **A10** asserts that order does not matter in both conjoining and disjoining a proposition B with a proposition A, **A11** – **A14** provide analogues of **#Dist1** and **#Dist2** for disjunctive-parthood and conjunctive-parthood. Rather than offering an intuitive basis for accepting the principles above, I will take **A10** – **A14** to be justified by their $\mathcal{N}$-validity given the reasons presented above for adopting the inclusive semantics along with the full range of normal models included in $\mathcal{N}$. It remains, however, to provide a broader description of the $\mathcal{N}$-logical consequence relation, surveying the space of $\mathcal{N}$-valid principles. In the following section, I will make a start by providing a logic for propositional identity which is sound over $\mathcal{N}$ given the inclusive semantics.

5 A Logic of Propositional Identity

Whereas §2 considered a range of principles in which identity operators occurred within the scope of extensional operators, the syntax presented in §3 restricted consideration to the propositional identity sentences in $\texttt{id}(\mathbb{L})$. Imposing this restriction simplified the semantics, while still providing a systematic means of evaluating identity sentences for validity. Given these syntactic restrictions, I will refer to the axiom system which results from combining the axioms **A1** – **A12** with the following rules of inference as *The First-Degree Logic for Propositional Identity* (PI^1), where **A13** and **A14** may then be derived:

R1 $A \equiv B \vdash B \equiv A$.

R2 $A \equiv B \vdash (A \wedge C) \equiv (B \wedge C)$.

R3 $A \equiv B \vdash \neg A \equiv \neg B$.

R4 $A \equiv B \vdash (A \vee C) \equiv (B \vee C)$.

R5 $A \equiv B,\ B \equiv C \vdash A \equiv C$.

I will take $\vdash^1_{\text{PI}}$ to be the smallest relation to satisfy the axioms and rules of inference for PI^1 which is closed under the standard structural rules, where $\varphi \in \texttt{id}(\mathbb{L})$ is a *theorem* of PI^1 just in case $\vdash^1_{\text{PI}} \varphi$ as usual. It is straightforward to show that PI^1 is sound with respect to the inclusive semantics and class $\mathcal{N}$ of normal models of $\mathcal{L}$. Accordingly, the motivation presented above for adopting the inclusive semantics and class of models $\mathcal{N}$ extends to each of the theorems of PI^1. Additionally, **R1** – **R5** may be derived from **Ref**, **Imps**, and **Func** in a background classical propositional logic, providing further reason to accept these rules of inference as uncontroversial.

In addition to deriving **A13** and **A14** in PI^1, we may show more generally that PI^1 has the following duality property, where $\delta(\varphi)$ is the result of swapping all conjunction and disjunction signs in φ, and $\delta(\Gamma) = \{\delta(\gamma) : \gamma \in \Gamma\}$:

Duality: A logic Λ is *dual iff* $\delta(\Gamma) \vdash_\Lambda \delta(\varphi)$ whenever $\Gamma \vdash_\Lambda \varphi$.

In contrast to PI^1, Correia and Skiles (2019) present a logic of generalised identity (GI), which is the result of both excluding **R3** from PI^1 while also including **#Dist2**. Although Correia and Skiles explicitly exclude **#Dist1** from GI— thereby giving up duality— they do not offer any motivation for leaving **#Dist1** out when **#Dist2** has been included in GI. Instead, Correia and Skiles defer to the *supersentence semantics* provided in Correia (2016) over which **#Dist2** may be shown to be valid, despite admitting counterexamples to **#Dist1**. Although Correia says nothing to motivate the use of his semantics either by reference to a stock of principles which his semantics validates or else by means of an intended model, he nevertheless shows that his semantics is equivalent to a Finean inclusive state semantics for the extensional operators together with the following alternative clause for propositional identity:

$(\equiv)_\text{C}$ $\mathcal{M} \vDash_\text{C} A \equiv B$ *iff* $|A|^+ = |B|^+$.

Whereas $(\equiv)$ required A and B to have the same exact verifiers and falsifiers in $\mathcal{M}$, Correia only requires A and B to have the same exact verifiers in $\mathcal{M}$, taking $\varphi \in \texttt{id}(\mathbb{L})$ to be valid just in case $\mathcal{M} \vDash_\text{C} \varphi$ for all $\mathcal{M} \in \mathcal{N}^+$— i.e., $\vDash^{\mathcal{N}^+}_\text{C} \varphi$— where $\mathcal{N}^+$ is the class of nonvacuous normal models of $\mathcal{L}$.[30] Correia (2016, p. 109) does not, however, provide any explicit motivation for adopting $(\equiv)_\text{C}$ over the semantics given in $(\equiv)$, nor for restricting consideration to nonvacuous

[30] Fine and Jago (2019, p. 539) present a system of exact entailment where exact entailment is defined solely in terms of exact verification with corresponding implications for distribution, where $A \vee (B \wedge C)$ exactly entails $(A \vee B) \wedge (A \vee C)$ but not *vice versa*, while $A \wedge (B \vee C)$ and $(A \wedge B) \vee (A \wedge C)$ exactly entail each other, disrupting an otherwise natural duality.

models, despite how much turns on these choices. Having already considered the demerits of restricting consideration to nonvacuous models, I will focus on the results of adopting Correia's semantics for propositional identity.

If $\mathbb{P}_{\mathcal{S}}$ is to model the space of propositions expressed by the sentences of a language, it is natural to assume that $A \equiv B$ is true in an $\mathcal{S}$-model just in case A and B are assigned to the same object inside $\mathbb{P}_{\mathcal{S}}$ by that model. Indeed, this is precisely what $(\equiv)$ asserts. By contrast, $(\equiv)^+$ only requires two sentences to have the same exact verifiers for identity to hold, and so $A \equiv B$ may be true in a model for Correia despite the fact that $|A| \neq |B|$. This leads to a number of undesirable effects, perhaps most vividly exhibited by the invalidity of **R3**. In particular, counterexamples to **R3** may be naturally extended to counterexamples to **Func** from which it follows that the negation operator is opaque, and so it follows by **P2** that **LL** does not hold.[31] Despite these surprising results, Correia and Skiles say nothing to acknowledge or defend the conclusion that negation is an opaque operator. By contrast, I will take the extensional operators '$\neg$', '$\wedge$', and '$\vee$' to be paradigm cases of transparent operators, maintaining a classical reading of '$\rightarrow$' in **Func** and **LL**.[32]

The fact that Correia and Skiles cannot claim that **Func** and **LL** hold without exception in a language with operators for propositional identity and the extensional operators is a direct result of the asymmetry between truth and falsity which Correia (2016) includes in his semantics. By contrast, it is easy to see why **R3** comes out valid when one takes both the truth-conditions and falsity-conditions for sentences into consideration in evaluating propositional identity claims as in $(\equiv)$ from §3. Assuming $A \equiv B$ is true in an arbitrary model $\mathcal{M}$, it follows from $(\equiv)$ that A and B have the same exact verifiers and falsifiers. Given that negation inverts the exact verifiers and falsifiers, $\neg A$ and $\neg B$ will also have the same exact verifiers and falsifiers, and so $\neg A \equiv \neg B$ will come out true in $\mathcal{M}$. Of course, this argument breaks down if the truth of $A \equiv B$ only requires the sameness of the truth-conditions. As brought out by $\mathcal{M}_D$ above, two sentences may share the same truth-condition without sharing the same falsity-condition, and so the fact that A and B share the same truth-condition says nothing of whether $\neg A$ and $\neg B$ share the same truth-condition. Despite limiting consideration to truth-conditions in evaluating propositional identity claims, Correia maintains consideration of both truth-conditions and falsity-conditions in evaluating the extensional sentences in $\texttt{ext}(\mathbb{L})$, making it all the more mysterious what motivates the sudden asymmetry late in his semantics.[33]

[31] Similar problems arise for Correia's (2004) semantics of analytic containment.

[32] Other transparent operators include the metaphysical modals '$\Box$' and '$\Diamond$', constitutive operators for essence and ground '$\sqsubseteq$' and '$\leqslant$', as well as the propositional identity operator '$\equiv$' itself. I take it that metaphysics ought to concern itself with working out which of the principles that can be articulated with such operators as these hold in full generality.

[33] Although one could define $A \equiv B := (A \equiv_{\text{G}} B) \wedge (\neg A \equiv_{\text{G}} \neg B)$ in a less syntactically restricted language, or uniquely characterise $\equiv$ with the rules $A \equiv_{\text{G}} B, \neg A \equiv_{\text{G}} \neg B \vdash A \equiv B$, $A \equiv B \vdash A \equiv_{\text{G}} B$, and $A \equiv B \vdash \neg A \equiv_{\text{G}} \neg B$, one must ask why $\equiv_{\text{G}}$ has be axiomatised rather than propositional identity $\equiv$. Correia and Skiles (2019) do not, however, consider either this definition or this characterisation of $\equiv$ in terms of the rules previously indicated.

Rather than maintaining an asymmetry between the truth-conditions and falsity-conditions in the semantics for propositional identity, I will take the validity of **R3** to be an immediate consequence of ($\equiv$) together with $(\neg)^+$ and $(\neg)^-$, from which it follows that negation is a transparent operator.

6 Subject-Matter Revisited

Recall from §2 the manner in which differences in subject-matter were taken to indicate differences between propositions. In particular, **#Necs**, **#Imps**, **#Abs1**, and **#Abs2** were all found to admit of exceptions, both motivating and constraining the development of the state semantics for $\mathcal{L}$ presented in §3. I then argued in §4 that there are strong abductive reasons for taking exception to **#Dist1** and **#Dist2**, where §5 presented a logic of propositional identity which is sound over the semantics. It remains, however, to extend the language $\mathcal{L}$ to include the subject-matter operator 'σ', providing both a semantics and logic for 'σ', thereby supplying a more substantial theory of subject-matter than the collection of principles initially presented in §2.

We may begin by considering the extension $\mathcal{L}^\sigma = \langle \mathbb{L}, \neg, \sigma, \vee, \wedge, \equiv \rangle$ of $\mathcal{L}$, defining the *pre-identity sentences* of $\mathcal{L}^\sigma$ as follows, where $p \in \mathbb{L}$ is arbitrary:

$$A ::= p \mid \neg A \mid \sigma A \mid A \wedge A \mid A \vee A.$$

Letting $\texttt{pid}(\mathbb{L})$ be the set of pre-identity sentences of $\mathcal{L}^\sigma$, I will take $A \equiv B$ to be an *identity sentence* in $\mathcal{L}^\sigma$ for any $A, B \in \texttt{pid}(\mathbb{L})$, where $\texttt{id}^\sigma(\mathbb{L})$ is the set of all identity sentences in $\mathcal{L}^\sigma$, and $\texttt{eq}(\mathbb{L}) \subseteq \texttt{id}^\sigma(\mathbb{L})$ is the set of all *equivalences* in $\mathcal{L}^\sigma$ of the form $A \Leftrightarrow B$. In addition to maintaining the inclusive semantics defended in §3 together with the clause ($\equiv$) for propositional identity, we may now consider the following clauses for the subject-matter operator:

$(\sigma)^+_{\text{C}}$ $\mathcal{M}, s \Vdash \sigma A$ *iff* $x \sqsubseteq s \sqsubseteq y$ for some $x, y \in S$ where $\mathcal{M}, x \Vdash A$ and $\mathcal{M}, y \Vdash A$.

$(\sigma)^-_{\text{C}}$ $\mathcal{M}, s \dashv\mid \sigma A$ *iff* $x \sqsubseteq s \sqsubseteq y$ for some $x, y \in S$ where $\mathcal{M}, x \dashv\mid A$ and $\mathcal{M}, y \dashv\mid A$.

A set of states X is *convex* just in case $y \in X$ whenever $x \sqsubseteq y \sqsubseteq z$ for some $x, z \in X$. Letting $\lceil X \rceil = \{y : x, z \in X \text{ and } x \sqsubseteq y \sqsubseteq z\}$ be the *convex closure* of X, it is easy to show that $\lceil \lceil X \rceil \rceil = \lceil X \rceil$.[34] We may then refer to the semantic clauses given above as the *convex semantics* for σ since the exact verifiers (falsifiers) for σA is the convex closure of the exact verifiers (falsifiers) for A. Letting PI^1_{C} be the result of including the following axioms and rule of inference in PI^1, we may show that given the convex semantics, PI^1_{C} is sound over the class of nonvacuous normal models $\mathcal{N}^+$ of $\mathcal{L}^\sigma$:

I am grateful to Kit Fine and Tim Williamson for helpful discussion of these points.

[34] Alternatively, one could define $\lceil X \rceil = [X, X]$, where $[X]$ is defined in *Span* from §3 above.

R6 $A \equiv B \vdash A \Leftrightarrow B$

S8 $\sigma\sigma A \equiv \sigma A.$

S9 $A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$

S10 $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$

It is worth noting that **R6** makes 'σ' a transparent operator such that σ operates solely on the propositions expressed by the sentences to which 'σ' is appended, thereby capturing **Obj** given in §2. Additionally, **S3** – **S7** stated in §2 follow immediately from **R6** given the identities included in PI^1.

Were one to extend consideration to all normal models in $\mathcal{N}$, both **S9** and **S10** will admit of counterexamples. Letting $\mathcal{M}_F = \langle S_F, \subseteq, |\cdot|_F \rangle$ with $S_F = \mathcal{P}(\{a, b, c, d, e, f\})$ where $|p_1|_F = \langle \{\{a\}\}, \{\{b\}\} \rangle$, $|p_2|_F = \langle \{\{c\}\}, \{\{d\}\} \rangle$, $|p_3|_F = \langle \varnothing, \{\{e\}\} \rangle$, and $|p_4|_F = \langle \{\{f\}\}, \varnothing \rangle$, for pairwise distinct a, b, c, d, e, and f, we may derive the following identities:

$$
\begin{aligned}
|p_1 \vee (p_2 \wedge p_3)|_F &= \langle \underline{\{\{a\}\}}, \{\{b,d\}, \{b,e\}, \{b,d,e\}\} \rangle \\
|(p_1 \vee p_2) \wedge (p_1 \vee p_3)|_F &= \langle \underline{\{\{a\}, \{c\}, \{a,c\}\}}, \{\{b,d\}, \{b,e\}, \{b,d,e\}\} \rangle \\
|p_1 \wedge (p_2 \vee p_4)|_F &= \langle \{\{a,c\}, \{a,f\}, \{a,c,f\}\}, \underline{\{\{b\}\}} \rangle \\
|(p_1 \wedge p_2) \vee (p_1 \wedge p_4)|_F &= \langle \{\{a,c\}, \{a,f\}, \{a,c,f\}\}, \underline{\{\{b\}, \{d\}, \{b,d\}\}} \rangle.
\end{aligned}
$$

Since the convex closures of the underlined sets are not identical, neither **S9** nor **S10** is $\mathcal{N}$-valid on the convex semantics. Although one could rule out all such counterexamples by restricting consideration to the nonvacuous models in $\mathcal{N}^+$, imposing such restrictions faces the same criticism brought out in §4. Moreover, without justifying the restriction to $\mathcal{N}^+$, the convex semantics fails to provide a basis of support upon which to claim that the distribution laws respect sameness of subject-matter as asserted by **S9** and **S10**.

Despite the shortcomings faced by the convex semantics, we may observe that '$\Leftrightarrow$' has the same derived semantic clause as the semantics which Fine (2016) provides for *synonymy* in the first degree fragment of Angell's (1989) logic of analytic containment (AC), or what Fine calls *analytic equivalence*, where $\lceil A \rceil^{\pm} = \{y : x \sqsubseteq y \sqsubseteq z \text{ and } x, z \in |A|^{\pm}\}$ is the convex closure of $|A|^{\pm}$:

($\Leftrightarrow$) $\mathcal{M} \vDash A \Leftrightarrow B$ *iff* $\lceil A \rceil^{\pm} = \lceil B \rceil^{\pm}$.

Whereas synonymy asserts the identity of the meanings of two terms— in this case sentences— the same cannot be said for identity of subject-matter. In particular, both **S1** and **S2** reproduced below ought to come out valid:

S1 $\neg A \Leftrightarrow A$

S2 $A \wedge B \Leftrightarrow A \vee B$

Although $\neg A$ and A may be said to have the same subject-matter, they do not have the same meaning, where something similar may be said of $A \wedge B$ and $A \vee B$. However, neither **S1** nor **S2** is valid over the convex semantics. Not only do these considerations provide reason to reject the convex semantics for subject-matter, they also raise doubts for Angell's (1989) logic AC. Given that synonymy in AC is coextensive with sameness of subject-matter in PI^1_C, and that distinct propositions may nevertheless have the same subject-matter, it follows that synonymy in AC is not as fine-grained as propositional identity. In particular, we may observe that although **S9** and **S10** have been included in PI^1_C, neither **#Dist1** nor **#Dist2** belong to PI^1_C. However, this is far from natural. Rather, one may expect synonymy to be at least as fine-grained as propositional identity, if not much more fine-grained on account of the same propositions being expressed by sentences with different meanings.[35]

Despite the disparity between the theoretical targets for analytic equivalence and propositional identity, it is worth reviewing Fine's reasons for taking the identity of the convex closures of the exact verifiers (falsifiers) for A and B to provide a semantics for analytic equivalence rather than the identity of sets of exact verifiers (falsifiers) for A and B. To begin with, Fine defines the following notion of containment where T and U are arbitrary sets of states:

Containment: $T < U$ *iff*: (1) for all $u \in U$, there is some $t \in T$ where $t \sqsubseteq u$; and
(2) for all $t \in T$, there is some $u \in U$ where $t \sqsubseteq u$.

Instead of providing an independent theoretical target for containment, Fine (2016) takes containment to provide a semantics for a *unilateral* notion of analytic entailment which only concerns exact verifiers, writing:

> Containment is the relation between contents which is the analogue of the relation of analytic entailment between statements. Thus we will want to say that A analytically entails C just in case the content of A contains the content of C. (p. 207) [...] If the relation $T < U$ is genuinely to represent a relation of partial content, of T being *part* of the content U, then we would expect the relation to be antisymmetric. (p. 208)

Even in supposing there to be a clear theoretical target by which to evaluate accounts of analytic entailment, Fine says nothing in support of the claim that containment amounts to the most natural notion of partial content, or that it provides the most natural semantics for analytic entailment. Nevertheless, it is clear that Fine takes antisymmetry to be essential to any genuine parthood relation, and so given this assumption, antisymmetry is required of containment insofar as containment is to be a notion of partial content. Although Fine (2016, p. 208) shows that containment is only antisymmetric for convex sets of states, he provides no other grounds for restricting attention to convex sets of states rather than revising his definition of containment.

Fine (2016, p. 201) defines analytic entailment as $A \rightarrow B := A \leftrightarrow A \wedge B$, where '$\leftrightarrow$' expresses analytic equivalence. Although Fine takes '$\leftrightarrow$' to be a primitive term instead of being defined in terms of 'σ' as above, the semantics that Fine (p. 217) provides for '$\leftrightarrow$' is equivalent to ($\leftrightarrow$) derived above. We may

[35] Fine (2016) proves that AC is sound and complete over his semantics, and so $\varphi \in \mathtt{eq}(\mathbb{L})$ is a theorem of PI^1_C just in case φ is valid over $\mathcal{N}^+$ given the convex semantics.

then follow Fine in deriving the following semantic clause from the definitions of analytic equivalence and containment for all $\mathcal{M} \in \mathcal{N}^+$:

($\rightarrowtriangle$) $\mathcal{M} \vDash A \rightarrowtriangle B$ *iff* $\lceil B \rceil^+ < \lceil A \rceil^+$ and $\lceil B \rceil^- \subseteq \lceil A \rceil^-$.[36]

Given any A which has exact verifiers, and any B which does not have exact verifiers, $A \wedge B \rightarrowtriangle A$ will not hold, contrary to Fine's (p. 201) expectations. However, following Fine (p. 205) in restricting attention to nonvacuous models faces the same objections brought out in §4 above. By contrast, one may take conjunctive-parthood to provide a notion of partial content where $\ulcorner A \sqsubseteq B \urcorner$ reads $\ulcorner A$ is analytically entailed by $B \urcorner$, observing that $A \sqsubseteq A \wedge B$ and $B \sqsubseteq A \wedge B$ are valid even without restricting consideration to nonvacuous models. Although it remains to specify a theoretical target for partial content against which conjunctive-parthood may be evaluated for adequacy, we may derive the following results, where I have included disjunctive-parthood for comparison:

($\sqsubseteq$) $\mathcal{M} \vDash A \sqsubseteq B$ *iff* $|A| \sqsubseteq |B|$.

($\leqslant$) $\mathcal{M} \vDash A \leqslant B$ *iff* $|A| \leqslant |B|$.[37]

In addition to these results above, conjunctive-parthood may be shown to be antisymmetric so long as **A1**, **A3**, and **R5** all hold.[38] Accordingly, taking conjunctive-parthood to play the role of partial content avoids the need to close the exact verifiers and falsifiers for the propositions in question under convexity. Without an argument that ($\rightarrowtriangle$) is preferable to ($\sqsubseteq$) in attempting to provide an adequate semantics for analytic entailment, the assumption that partial content ought to be antisymmetric does not provide a reason to require propositions to be convex, for as we have seen, conjunctive-parthood is antisymmetric even without closing the exact verifiers and falsifiers under convexity.

Setting aside the connections between sameness of subject-matter in PI^-_{C} and analytic equivalence in AC, there is good reason to consider the *undirected semantics* for 'σ' on account of validating **S1** over all normal models in $\mathcal{N}$:

$(\sigma)^+_{\text{U}}$ $\mathcal{M}, s \Vdash \sigma A$ *iff* $s \sqsubseteq t$ for some $t \in S$ where $\mathcal{M}, t \Vdash A \vee \neg A$.

$(\sigma)^-_{\text{U}}$ $\mathcal{M}, s \dashv\!\vert\, \sigma A$ *iff* $s \sqsubseteq t$ for some $t \in S$ where $\mathcal{M}, t \dashv\!\vert\, A \wedge \neg A$.

On the undirected semantics, σA is both exactly verified and falsified by the parts of the exact verifiers and falsifiers for A. It also follows that both of the following principles are valid, where **S12** follows from **S1** and **S11** by **R5**:

[36] Whereas Fine (2016, p. 217) considers the wider space of nonvacuous models where sentence letters may be assigned to propositions whose contents are nonempty but not necessarily closed under fusion, it will suffice for present purposes to restrict consideration to $\mathcal{N}^+$.

[37] It is worth observing that the semantics for disjunctive-parthood is the result of inverting the exact verifiers and falsifiers in the semantics for conjunctive-parthood, thereby validating $A \sqsubseteq B \vdash \neg A \leqslant \neg B$ and $A \leqslant B \vdash \neg A \sqsubseteq \neg B$. See Fine (2017a, p. 661) for related results.

[38] *Proof:* If $A \sqsubseteq B$ and $B \sqsubseteq A$, then $A \wedge B \equiv B$ and $B \wedge A \equiv A$, and so $A \equiv A \wedge B$ by **A3** and **R5**. Thus $A \equiv B$ again by **R5**, where the converse is immediate from **A1**. ☐

S11 $\sigma A \equiv \neg\sigma A$. **S12** $\sigma\neg A \equiv \neg\sigma A$.

Rather than taking 'σ' to be a sentential operator, Fine (2016, 2017b, 2020) identifies the subject-matter of a proposition with the fusion of all exact verifiers and falsifiers for that proposition. Accordingly, the subject-matter of A is not a proposition at all for Fine, but rather a single state, and so neither **S11** nor **S12** can be interpreted on Fine's semantics for subject-matter. Despite the disparity in kind between Fine's objectual account of subject-matter and the propositional account assumed above, the undirected semantics validates the same equivalences in $\mathtt{eq}(\mathbb{L})$ as Fine's account of subject-matter.[39] In particular, **S2** is invalid over the class of normal models $\mathcal{N}$, contrary to the expectations above. For instance, if A has no exact verifiers, then neither will $A \wedge B$, though $A \vee B$ will retain all of the exact verifiers for B, where a similar discrepancy may occur if B has exact falsifiers but A does not. Thus the parts of the exact verifiers and falsifiers for $A \wedge B$ may diverge from the parts of the exact verifiers and falsifiers for $A \vee B$, resulting in counterinstances to **S2**.

Rather than restricting attention to the nonvacuous models of $\mathcal{L}$, or else excluding **S2** from the logic for subject-matter, I will provide a semantics which validates all of the subject-matter principles considered above. It will help to begin with a propositional analogue of another idea that Fine (2016, 2017b, 2020) develops, focusing at first on the *positive subject-matter* operator 'σ^+'. In contrast to Fine who takes $\sigma^+ A$ to be the fusion of the exact verifiers for A, I will draw on the duality operator 'δ' defined in §5 in order to provide the following *dual semantics* for the positive subject-matter operator 'σ^+', where $\ulcorner\sigma^+ A\urcorner$ reads ⌜It is partially the case that A⌝:[40]

$(\sigma^+)^+_{\mathrm{D}}$ $\mathcal{M}, s \Vdash \sigma^+ A$ *iff* $s \sqsubseteq t$ for some $t \in S$ where $\mathcal{M}, t \Vdash A \vee \delta(A)$.

$(\sigma^+)^-_{\mathrm{D}}$ $\mathcal{M}, s \dashv\!\mid \sigma^+ A$ *iff* $s \sqsubseteq t$ for some $t \in S$ where $\mathcal{M}, t \dashv\!\mid A \wedge \delta(A)$.

Were we to omit '$\delta(A)$' from the clauses above, $\sigma^+(A \wedge B)$ and $\sigma^+(A \vee B)$ will fail to be identical in cases where A has exact verifiers but B does not. However, given the semantics for conjunction and disjunction, it is natural to expect that it being partially the case that $A \wedge B$ is the same as it being partially the case that $A \vee B$. A similar point holds for the *negative subject-matter* operator which we may define by $\sigma^- A := \sigma^+\neg A$, where $\ulcorner\sigma^- A\urcorner$ reads ⌜It is partially not the case that A⌝. By contrast, Fine's objectual account takes $\sigma^- A$ to be the fusion of all exact falsifiers for A. Accordingly, Fine must restrict consideration to just the nonvacuous models of $\mathcal{L}$, at least insofar as $A \wedge B$ and $A \vee B$ are to have the same positive and negative subject-matters in common.

Given the reading of 'σ^+' together with the definition of 'σ^-', we may define the subject-matter operator by $\sigma A := \sigma^+ A \vee \sigma^- A$, thereby incorporating the

[39] See Fine (2016, p. 209) and Hawke (2018, p. 718) for similar observations.

[40] I will assume an improper reading of 'partially' so that it being the case that A entails that it is partially the case that A. See also Fine's (2017b, p. 699) notion of partial aboutness.

central idea behind the undirected semantics into a dual semantics for 'σ'. Thus ⌜σA⌝ reads ⌜It is partly the case that A or it is partly not the case that A⌝, where we may then derive the following dual semantic clauses for 'σ':

$(\sigma)^+_{\text{D}}$ $\mathcal{M}, s \Vdash \sigma A$ *iff* $s \sqsubseteq t$ for some $t \in S$ where $\mathcal{M}, t \Vdash A \vee \neg A \vee \delta(A) \vee \delta(\neg A)$.

$(\sigma)^-_{\text{D}}$ $\mathcal{M}, s \dashVv \sigma A$ *iff* $s \sqsubseteq t$ for some $t \in S$ where $\mathcal{M}, t \dashVv A \wedge \neg A \wedge \delta(A) \wedge \delta(\neg A)$.

That the dual semantics validates both **S1** and **S2** over $\mathcal{N}$ should not surprise, for the exact verifiers for σA include all parts of the exact verifiers for A, $\neg A$, and their duals, where the same may be said of the exact falsifiers. Additionally, the dual semantics validates the following distribution laws:

S13 $\sigma(A \vee B) \equiv (\sigma A \vee \sigma B)$. **S14** $\sigma(A \wedge B) \equiv (\sigma A \wedge \sigma B)$.

Instead of revising the syntax for $\mathcal{L}^\sigma$, we may continue to take 'σ' to be primitive for present purposes, excluding 'σ^+' and 'σ^-' from $\mathcal{L}$. Nevertheless, we may draw on the definition of subject-matter given above in order to justify the informal reading of 'σ' which the dual semantics makes precise. Additionally, we may extend the proof theory to accommodate the subject-matter operator by letting PI^1_σ extend PI^1 to include **R6** along with **S1**, **S2**, **S8**, **S11**, and **S13**, where we may then derive **S3** – **S7**, **S9**, **S10**, **S12**, and **S14**.[41]

Having begun to survey the space of subject-matter principles, it remains to provide an account of relevance. Were one to take the relevance operator '$\preceq$' to be primitive, **Rel** could be captured by adding the following rule of inference to PI^1_σ, where **L1** – **L4** could also be included in the logic:

R7 $A \Leftrightarrow B,\ C \preceq A \vdash C \preceq B$.

A natural motivation for **R7** conceives of relevance as a parthood relation for subject-matter, so that $A \preceq B$ asserts that the subject-matter of A is part of the subject-matter of B.[42] Thus, if A and B share the same subject-matter, where the subject-matter of C is part of the subject-matter of A, then the subject-matter of C is also part of the subject-matter of B. This justification for **R7** can be strengthened by defining relevance as $A \preceq B := \sigma A \leqslant \sigma B$, or equivalently, as $A \preceq B := \sigma A \sqsubseteq \sigma B$. Informally, A is relevant to B just in case the subject-matter of A is a disjunctive-part of the subject-matter of B, or equivalently, the subject-matter of A is a conjunctive-part of the subject-matter of B. Regardless of which convention one adopts, **R7** along with **L1** – **L4** may be derived in PI^1_σ, where **Obj** and **Rel** follow from **R6** and **R7** by classical reasoning, thereby providing the beginnings of a theory of relevance.

[41] It is worth noting that every $\varphi \in \mathtt{eq}(\mathbb{L})$ is *self-dual* insofar as $\varphi \vdash^1_{\text{PI}_\sigma} \delta(\varphi)$.

[42] By contrast, there are the directed notions of *positive relevance* $A \preceq^+ B := \sigma^+ A \leqslant \sigma^+ B$ and *negative relevance* $A \preceq^- B := \sigma^- A \leqslant \sigma^- B$ (or equivalently, $A \preceq^- B := \sigma^+ A \sqsubseteq \sigma^+ B$) which do not assimilate what is relevant to A and what is relevant to $\neg A$. For instance, as brought out in footnote 18, although the exact verifier states for a sentence must be wholly relevant to that sentence, those states may fail to be wholly relevant to its negation.

It is worth noting that relevance is not an entailment relation on the present understanding. For instance, although both A and B are relevant to $A \wedge B$ by **L2** and **L4**, neither A nor B entails $A \wedge B$. Accordingly, the present account of relevance is not to be assimilated to the notions of relevant entailment, or analytic implication, as developed by the relevance logicians.[43] Although one could attempt to draw on a theory of relevance in combination with an account of implication in order to provide an analysis of a relevant implication relation, we may observe that disjunctive-part and conjunctive-part already amount to forms of relevant implication. In particular, we may derive the following:

T1 $A \leqslant B \vdash A \preceq B$. **T2** $A \sqsubseteq B \vdash A \preceq B$.

If $A \leqslant B$, then by definition $A \vee B \equiv B$, and so B obtains in any possibility in which A obtains. Thus $A \leqslant B$ may be taken to be an implication relation from A to B, where the former must be relevant to the latter by **T1**. By contrast, we may observe that if $A \sqsubseteq B$, then by definition $A \wedge B \equiv B$, and so A obtains in every possibility in which B obtains. Accordingly, $A \sqsubseteq B$ may be taken to be an implication relation from B to A, where the latter must be relevant to the former by **T2**. Given that both relevance and implication flow from left to right for disjunctive-part, but in opposite directions for conjunctive-part, it is disjunctive-part rather than conjunctive-part which makes for the most apt comparison with the entailment relations studied in relevance logics.

In contrast to the rough outline presented in §2, the axioms and rules included in PI^1_σ provide a much richer theory of both subject-matter and relevance. Nevertheless, it remains to establish completeness over the semantics defended above, if the semantics has a complete logic at all. Additionally, it is desirable to provide a semantics that does not restrict consideration to the sentences in $\mathtt{id}^\sigma(\mathbb{L})$, interpreting sentences which contain any combination of sentential operators included in the language.[44] Nevertheless, maintaining the restriction to $\mathtt{id}^\sigma(\mathbb{L})$ provides a first step towards a logic of propositional identity with greater expressive power. In particular, I will take PI^1_σ to provide an elegant theory of propositional identity with a well motivated semantics which is able to individuate propositions according to their subject-matter rather than their modal profile alone, thereby satisfying the aims set out above.

7 Appendix

I will begin by considering propositional languages of the form $\mathcal{L} = \langle \mathbb{L}, \vec{\mathcal{Q}} \rangle$ where each $\mathcal{Q}^n_i \in \vec{\mathcal{Q}}$ is an n-ary sentential operator for some $n \in \mathbb{N}$, and $\vec{\mathcal{Q}}$ includes the extensional connectives '$\neg$', '$\wedge$', and '$\vee$' along with the propositional identity operator '$\equiv$'. The *well-formed sentences* in $\mathtt{wfs}(\mathbb{L})$ consist of the sentence letters in $\mathbb{L}$ along with any $\mathcal{Q}^n_i(\vec{O})$, where $\mathcal{Q}^n_i \in \vec{\mathcal{Q}}$ and $\vec{O}$ is a sequence of n

[43] For instance, see Anderson et al. (1976), Angell (1989), and Parry (1989).
[44] I provide such a semantics in Brast-McKie (2020).

well-formed sentences. Assuming a background classical logic in which $\ulcorner A \to B \urcorner$ abbreviates $\ulcorner \neg A \vee B \urcorner$, we may consider the following:

Ref $A \equiv A$.

Sym $(A \equiv B) \to (B \equiv A)$.

LL $(A \equiv B) \to (C \to C_{(B/A)})$.

Trans $(A \equiv B) \to [(B \equiv C) \to (A \equiv C)]$.

Imps $(A \equiv B) \to (A \to B)$.

Func $(A \equiv B) \to [\mathcal{Q}(\vec{O}) \equiv \mathcal{Q}(\vec{O}_{(B/A)})]$.

As above, $\mathcal{L}$ is *transparent* just in case all instances of **Func** hold, where $\ulcorner \mathcal{Q}(\vec{O}_{(B/A)}) \urcorner$ is the result of replacing one or more instances of $\ulcorner A \urcorner$ which occur as members of the sequence $\ulcorner \vec{O} \urcorner$ with $\ulcorner B \urcorner$, where similarly $\ulcorner \mathcal{Q}(\vec{O}_{[B/A]}) \urcorner$ is the result of replacing all instances of $\ulcorner A \urcorner$ which occur as members of the sequence $\ulcorner \vec{O} \urcorner$ with $\ulcorner B \urcorner$. Additionally, I will take $\ulcorner C_{(B/A)} \urcorner$ to be the result of replacing one or more instances of $\ulcorner A \urcorner$ as it occurs anywhere in $\ulcorner C \urcorner$ with $\ulcorner B \urcorner$, as well as taking $\ulcorner C_{[B/A]} \urcorner$ to be the result of replacing all instances of $\ulcorner A \urcorner$ as it occurs anywhere in $\ulcorner C \urcorner$ with $\ulcorner B \urcorner$. We may now prove the following propositions.

P1 If $\mathcal{L}$ is transparent, then **Ref** and **Imps** entail both **Sym** and **Trans**.

Proof. Assuming that $\mathcal{L}$ is transparent, $(A \equiv B) \to [(A \equiv A) \equiv (B \equiv A)]$ follows, where $[(A \equiv A) \equiv (B \equiv A)] \to [(A \equiv A) \to (B \equiv A)]$ holds by **Imps**, and so $(A \equiv B) \to (B \equiv A)$ follows from **Ref** by propositional logic. Again by transparency, $(A \equiv B) \to ([(A \equiv C) \to (A \equiv C)] \equiv [(B \equiv C) \to (A \equiv C)])$, and so $(A \equiv B) \to ([(A \equiv C) \to (A \equiv C)] \to [(B \equiv C) \to (A \equiv C)])$ by **Imps**. Since $(A \equiv C) \to (A \equiv C)$ holds by propositional logic, we may conclude as desired that $(A \equiv B) \to [(B \equiv C) \to (A \equiv C)]$. □

L1 If $\mathcal{L}$ is transparent, then **Ref** and **Imps** entail $(A \equiv B) \to (C \equiv C_{[B/A]})$.

Proof. Assuming that $\mathcal{L}$ is transparent, the proof proceeds by induction on the complexity of $C \in \texttt{wfs}(\mathbb{L})$. Of course, if $C \in \mathbb{L}$, then $(A \equiv B) \to (C \equiv C_{[B/A]})$ holds by propositional logic if A occurs in C, and $C \equiv C_{[B/A]}$ holds by **Ref** otherwise, where $(A \equiv B) \to (C \equiv C_{[B/A]})$ follows by propositional logic.

Assume for induction that $(A \equiv B) \to (C \equiv C_{[B/A]})$ holds whenever $\texttt{comp}(C) \leqslant n$, and let $\texttt{comp}(C) = n + 1$. Assuming that $C = \mathcal{Q}^n(\vec{D})$, we may observe that for all $1 \leqslant i \leqslant n$ that $(A \equiv B) \to (D_i \equiv D_{i[B/A]})$ follows by hypothesis, where $(D_i \equiv D_{i[B/A]}) \to [\mathcal{Q}^n(\vec{E}) \equiv \mathcal{Q}^n(\vec{E}_{[D_{i[B/A]}/D_i]})]$ by the transparency of $\mathcal{L}$ for any $\vec{E}$. Assuming $A \equiv B$, it follows for all $1 \leqslant m \leqslant n$ that $\mathcal{Q}^n(\vec{D}_{[D_{1[B/A]}/D_1]\ldots[D_{m[B/A]}/D_m]}) \equiv \mathcal{Q}^n(\vec{D}_{[D_{1[B/A]}/D_1]\ldots[D_{m+1[B/A]}/D_{m+1}]})$. Given that **Ref** and **Imps**, it follows by **P1** that **Trans** holds, and so by $n-1$ applications of **Trans**, $\mathcal{Q}^n(\vec{D}) \equiv \mathcal{Q}^n(\vec{D}_{[D_{1[B/A]}/D_1]\ldots[D_{n[B/A]}/D_n]})$. We may then observe that $\mathcal{Q}^n(\vec{D}_{[D_{1[B/A]}/D_1]\ldots[D_{n[B/A]}/D_n]} = \mathcal{Q}^n(\vec{D})_{[B/A]}$, and so it follows by discharging our assumption that $(A \equiv B) \to (\mathcal{Q}^n(\vec{D}) \equiv \mathcal{Q}^n(\vec{D})_{[B/A]})$. Since $C = \mathcal{Q}^n(\vec{D})$, we may conclude that $(A \equiv B) \to (C \equiv C_{[B/A]})$. □

P2 Assuming **Ref** and **Imps**, then $\mathcal{L}$ is transparent just in case **LL** holds.

Proof. Assume **Ref** and **Imps**. Letting **LL** hold in $\mathcal{L}$ where $\mathcal{Q}$ is an operator in $\mathcal{L}$, it follows that $(A \equiv B) \to ([\mathcal{Q}(\vec{O}) \equiv \mathcal{Q}(\vec{O})] \to [\mathcal{Q}(\vec{O}) \equiv \mathcal{Q}(\vec{O})_{(B/A)}])$. However, given **Ref**, $\mathcal{Q}(\vec{O}) \equiv \mathcal{Q}(\vec{O})$, and so $(A \equiv B) \to [\mathcal{Q}(\vec{O}) \equiv \mathcal{Q}(\vec{O})_{(B/A)}]$. In particular, $(A \equiv B) \to [\mathcal{Q}(\vec{O}) \equiv \mathcal{Q}(\vec{O}_{(B/A)})]$ as in **Func**. Generalising on $A, B, \vec{O}$, and $\mathcal{Q}$, we may conclude that $\mathcal{L}$ is transparent.

Assume instead that $\mathcal{L}$ is transparent. Letting $p \in \mathbb{L}$ where p does not occur in B or C, it follows by **L1** that $(p \equiv B) \to (C_{(p/A)} \equiv C_{(p/A)[B/p]})$. However, $C_{(p/A)[B/p]} = C_{(B/A)}$, and so $(p \equiv B) \to (C_{(p/A)} \equiv C_{(B/A)})$. Thus $(A \equiv B) \to (C \equiv C_{(B/A)})$ in particular, and so **LL** follows by **Imps**. □

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