

# The Construction of Possible Worlds\*

*Benjamin Brast-McKie*

## Abstract

*Possible worlds* are often taken to be complete histories of everything. Insofar as there are *temporary sentences* that are true at some times and false at other times, evaluating a sentence at a possible world does not fix its truth-value. Moreover, if possible worlds are taken to be primitive, evaluating sentences at world-time pairs invalidates a perpetuity principle that what is necessarily the case is always the case where imposing model constraints cannot validate these principles without undermining the significance of the truth-conditions for the language. Rather, this paper takes *world states* to be maximal possible ways for things to be at an instant where the *task relation* encodes the possible transitions between world states. Possible worlds are then defined as functions from times to world states as constrained by the task relation. Since sentences are assigned truth-values at world states, times are exogenous to the truth-conditions for the language, eliminating unnecessary degrees of freedom from the definition of a model. By evaluating sentences at world-time pairs, the resulting semantic theory validates a logic for tense and modality in which the perpetuity principles are theorems, providing a logical foundation for reasoning about future contingency.

**Keywords:** Tense, Modality, Bimodal Logic, Task Semantics, Dynamical Systems

## 1 Introduction

Intensional semantic theories often conceive of possible worlds as complete histories of everything. Insofar as there are *temporary sentences* which are true at some times and false at others in the same possible world, the truth-value of a temporary sentence is not fixed by a possible world considered on its own. For instance, suppose that in a world  $w$ , the sentence ‘Cary is reading’ was true this morning but false in the afternoon. Merely specifying the world of evaluation  $w$  does not determine the truth-value of a temporary sentence, at least insofar as possible worlds are taken to be temporally extended histories rather than instantaneous world states.

---

<sup>0</sup>I am grateful to Miguel Buitrago, Bailey Fernandez, Justin Khoo, Graham Leach-Krouse, Jonas Werner, Steven Yablo, and the attendees of Minds & Metaphysics and the graduate seminar I taught on [The Modern History of Modal Logic](#) at MIT.

One response denies that there are temporary sentences, assuming instead that every sentence, perhaps implicitly, includes a reference to some time or other. An *eternalist* of this kind takes the sentence ‘Cary is reading’ to be incomplete and so to be replaced by the *permanent sentence* ‘Cary is reading at  $t$ ’ where  $t$  is a time.<sup>1</sup> However, including singular terms which refer to *times* makes an ontology of times a part of the topic of conversation. Although some sentences may be about certain times either imagined, fictional, experienced, theorized about, or otherwise discussed, the sentence ‘Cary is reading’ does not concern any time whatsoever, but rather has Cary and her engagement reading as the entire subject-matter. Moreover, since a permanent sentence that is true in  $w$  is true at any time in  $w$ , a permanent sentence  $\varphi$  is equivalent to the result of embedding  $\varphi$  under arbitrary tense operators, obviating the need to include tense operators in the language.<sup>2</sup> However, the truth-condition for ‘Cary is reading’ differs substantially from the result of embedding this sentence under tense operators. For instance, it might be true that Cary is reading, and yet false that she always has been reading, or is always going to be reading. Instead of excluding tense operators and temporary sentences such as ‘Cary is reading’ from the language, this paper provides a truth-conditional semantics which respects the natural morphology of both tensed and modal claims, describing their interactions in a bimodal language without positing unspoken references to an ontology of times.

Rather than including times in the logical form of the sentences of a language, the semantics pioneered by Montague [1] and Kaplan [2] evaluates sentences at world-time pairs. As a result, times are made endogenous to the interpretation of the language where sentences are assigned to sets of world-time pairs.<sup>3</sup> Given an *at least as early as* weak total order  $\leq$  on the times, this strategy provides a two-dimensional semantics for  $\Box$  and  $\Box$  which read ‘It has always been the case that’ and ‘It is always going to be the case that’, respectively. In particular,  $\Box\varphi$  is true at a world  $w$  and time  $x$  just in case  $\varphi$  is true at  $w$  and  $y$  for every time  $y < x$  where  $y < x := y \leq x \wedge x \not\leq y$  as usual. Similarly,  $\Box\varphi$  is true at a world  $w$  and time  $x$  just in case  $\varphi$  is true at  $w$  and  $y$  for every time  $y > x$ . As a result, it is easy to distinguish the truth-condition for ‘Cary is reading’ from the result of embedding this sentence under tense operators. Whereas ‘Cary is reading’ is assigned to the set of all world-time pairs in which Cary is reading in that world at that time, ‘Cary always was reading’ is assigned to the set of all world-time pairs where she is reading at all earlier times in that world. Something similar may be said for ‘Cary is always going to be reading’.

Defining  $\Diamond\varphi := \neg\Box\neg\varphi$  and  $\Diamond\varphi := \neg\Box\neg\varphi$ , the defined operators  $\Diamond$  and  $\Diamond$  express ‘It has been the case that  $\varphi$ ’ and ‘It is going to be the case that  $\varphi$ ’, respectively. In contrast to the eternalist, the two-dimensional semantics assigns distinct truth-conditions to temporary sentences of the form  $\varphi$ ,  $\Box\varphi$ ,  $\Box\varphi$ ,  $\Diamond\varphi$ ,  $\Diamond\varphi$  and sentences with iterated temporal operators. However, taking possible worlds to be structureless points within a model while nevertheless representing temporally extended histories comes at both an intuitive and theoretical cost. In particular, the following *perpetuity*

<sup>1</sup> Atomic permanent sentences could be regimented by ‘ $R_t(c)$ ’ or ‘ $R(c, t)$ ’. What matters is that the times are included in the logical form rather than as a parameter at which the sentence is evaluated.

<sup>2</sup> There are pathological cases if time is bounded, though these exceptions are beside the present point.

<sup>3</sup> Alternatively, sentences could be assigned to characteristic functions from world-time pairs to truth-values, or else functions from times (perhaps together with other contextual parameters) to functions from worlds to truth-values, all of which take times to be endogenous to the interpretation of the language.

*principles* are invalid where  $\triangle\varphi := \Box\varphi \wedge \varphi \wedge \Box\varphi$  and  $\nabla\varphi := \Diamond\varphi \vee \varphi \vee \Diamond\varphi$  may be read ‘It is always the case that  $\varphi$ ’ and ‘It is sometimes the case that  $\varphi$ ’ respectively and  $\Box$  and  $\Diamond$  are the metaphysical modals which are to be read as usual:

$$\mathbf{P1} \quad \Box\varphi \rightarrow \triangle\varphi.$$

$$\mathbf{P2} \quad \nabla\varphi \rightarrow \Diamond\varphi.$$

It is natural to assume that whatever is metaphysically necessary is always the case, or equivalently, whatever is sometimes the case is metaphysically possible. Insofar as metaphysical modality is the strongest objective modality, the semantic clauses for the metaphysical modals quantify over the broadest range of objective possibilities. If it is sometimes the case that  $\varphi$ , then there is a world and time in which  $\varphi$  is the case, and so it is metaphysically possible for  $\varphi$  to be the case as **P2** asserts. Since **P2** and **P1** are equivalent, these considerations justify **P1** with equal force.

As plausible as **P1** and **P2** may be, evaluating sentences at world-time pairs admits counterexamples to both of these principles. So long as sentences may be assigned to any set of world-time pairs and there is more than one time,  $\varphi$  may be true in every world  $w'$  at a time  $x$  without also being true in a world  $w$  at every time  $x'$ . Rather than weakening the logic by giving up the perpetuity principles, the following section will review three unsuccessful strategies for validating **P1** and **P2**. These considerations will motivate the semantics that I develop in §3 which validates these principles by defining possible worlds in terms of *world states*, *tasks*, and *times* rather than taking possible worlds to be primitive as traditionally assumed. After presenting the resulting logic along with a number of extensions, I will conclude in §4 by presenting an account of the openness of the future and drawing connections to dynamical systems theory. The formal results referred to throughout will be provided in §5.

## 2 Primitive Worlds

In order to provide a flexible semantics with which to characterize the modal systems that Lewis and Langford [3] first set out in Appendix II of their 1932 textbook, Kripke [4, 5] took the models of a modal language to include a nonempty set of *possible worlds*  $W$ , an *evaluation world*  $w$ , an *accessibility relation*  $R$ , and an *interpretation*  $\mathcal{I}$  assigning each sentence letter a truth-value at each world. Adding these resources improved on Carnap’s [6, 7] semantic theories that evaluated sentences at *state-descriptions* which, for any atomic sentence of the language, include that sentence or its negation but not both. Carnap [6] specified how state-descriptions are to be understood as follows:

A state description is a class of sentences which represents a possible specific state of affairs by giving a complete description of the universe of individuals with respect to all properties and relations designated by predicates in the system.<sup>4</sup> (p. 50)

Since Carnap’s state-descriptions are entirely syntactic in their construction and so determined by the primitive symbols included in a non-modal language, Carnap does

---

<sup>4</sup>In his earlier work, Carnap [8, p. 95] writes, “If a false sentence is not L-false, hence not self-contradictory, it describes a situation which [is] possible though not real,” and later that, “a system  $S$  has to do with many objects, and hence we have to consider the possible states of affairs of all the objects dealt with in  $S$  and with respect to all properties, relations, etc., dealt with in  $S$ ,” (p. 101) indicating a circumstantial reading of the modal rather than the interpretational modality that his theory was in a better position to support.

not provide a genuine model theory but rather a single structure by which to interpret a modal language. Building on Carnap's efforts, Kripke [9] first sought to evaluate sentences at *complete assignments* over a *domain*  $D$  where each assignment maps the singular terms to elements in  $D$  and  $n$ -adic predicates to sets of  $n$ -tuples of elements in  $D$  where sentence letters are assigned to truth or falsity. By contrast with Carnap's state-descriptions which belong to a single fully specified structure, Kripke took the domain  $D$  included in a model to be any set whatsoever, where it was by quantifying over all such models that Kripke defined validity for the language. Given that there are no two complete assignments that agree on all elements of the language, the range of complete assignments is determined by the primitive symbols included in the language together with the domain provided by the models of the language.

Despite the language relativity of Carnap's state-descriptions and the complete assignments that Kripke modeled after them, both of these constructions sought to represent *possibilities*. By quantifying over all or some possibilities, these accounts provided semantic clauses for the modal operators. Although the language relativity of the possibilities presents no issue for an interpretational modality that Carnap [6, 7] and Kripke [9] might be taken to have described, Kripke sought to accommodate a broader range of readings for the modal operators by decoupling the possibilities over which the modal operators quantified from the primitive symbols in the language.<sup>5</sup> Beginning with the propositional fragment, Kripke [4] introduced a primitive set of *possible worlds*  $W$  which he took to be any nonempty set whatsoever, where the interpretation of the language was handled by an independent *interpretation function* mapping each sentence letter and possible world to a truth-value.<sup>6</sup> By also specifying a primitive relation  $R$  over  $W$  for *relative possibility*, Kripke showed how to associate the reflexivity, symmetry, and transitivity constraints on  $R$  with the corresponding **T**, **B**, and **4** axioms which characterized the differences between the most prominent modal systems that Lewis and Langford had [3] had described. Although Kripke [5] went on to extend his semantics to include predicates and first-order quantifiers, it will suffice for present purposes to restrict consideration to the propositional fragment of the languages with which Carnap and Kripke were concerned.

Despite their differences, neither Carnap nor Kripke were focused on interpreting languages with tense operators or singular terms for an ontology of times.<sup>7</sup> Insofar as state-descriptions, complete assignments, and possible worlds are taken to determine the truth-values of temporary sentences, it is implausible to interpret these elements as temporally extended histories. Besides being irrelevant to the interpretation of a modal language without tense operators, taking possible worlds to be temporally extended leaves the truth-values for temporary sentences underdetermined, requiring the addition of a temporal parameter to the point of evaluation in order to specify the time in the possible world at which the sentence is to be evaluated. However,

---

<sup>5</sup>Kripke [10, 11] went on to describe a metaphysical reading of the modal operators for which it is inappropriate to relativize the range of possibilities to the expressive power of a language.

<sup>6</sup>Officially, Kripke [4] took the *models* on a *model structure*  $\langle W, R, w \rangle$  to be functions which take a propositional variable and world to a truth-value. I have replaced propositional variables with sentence letters and reserved the label 'model' to improve consistency with the definitions presented below.

<sup>7</sup>Although Kripke [12] interprets possible worlds as *moments* in a letter discussing Prior's [13] tensed interpretation the modal operators, he writes, "I myself was working with ordinary modal logic." For the published correspondence and accompanying discussion, see [14, p. 373].

neither Carnap nor Kripke provide any indication that times are to be considered in evaluating sentences. Since the possible worlds that Kripke took to be primitive cannot be interpreted as temporally extended, they must be instantaneous.<sup>8</sup>

Although it is natural to conceive of possible worlds as complete configurations of a system at a moment when interpreting the sentences of a language with either tense operators or modal operators but not both, the same cannot be said for bimodal languages with both tense and modal operators. Not only do instantaneous moments fail to specify what comes before or after when considered on their own, including an ordering of moments once and for all fails to capture the range of different orderings of moments that are possible. What is needed to interpret bimodal sentences is an encoding of both the modal and temporal dimensions of the semantics. To satisfy this demand, Montague [1] and Kaplan [2] developed extensions of Kripke’s semantics which assign truth-values to sentences at world-time pairs rather than at possible worlds on their own. In addition to a primitive set of worlds  $W$ , these theories include a primitive set of *times*  $T$  along with a weak total order  $\leq$  for the *at least as early as* relation.<sup>9</sup> Abstracting from their differences, I will take a *two-dimensional model*  $\mathcal{M}_2 = \langle W, T, \leq, |\cdot| \rangle$  to include an interpretation function where  $|p_i| \subseteq W \times T$  is a set of world-time pairs for each sentence letter  $p_i$  with  $i \in \mathbb{N}$ . Instead of taking the worlds in  $W$  to be instantaneous moments which do not specify a past or future, worlds may now be understood to be temporally extended histories. In order to avoid temporal ambiguity, sentences are evaluated at both a world and time, thereby indicating the moment in a world’s history at which a sentence is to be evaluated. For instance, although ‘Cary is reading’ may be true in  $w$  at  $x$ , the sentence may be false in  $w$  at other times, where it is for this reason that ‘Cary is reading’ is temporary.

However natural it may seem to extend Kripke’s semantics along the lines above, the resulting space of models invalidate **P1** and **P2**. Rather than constraining the space of models to exclude these counterexamples, Dorr and Goodman [15] follow Montague [1] in defending an *asymmetric semantics* which takes the modal operators to quantify over all world-time pairs while the temporal operators only quantify over times. After reviewing a number of shortcomings which discourage this approach in the following subsection, §2.2 will present the challenges that Kaplan’s [2] *symmetric semantics* for bimodal languages faces in attempting to validate the perpetuity principles. As I will show, constraining the space of models to exclude counterexamples to **P1** and **P2** while taking possible worlds to be primitive undermines the significance of the truth-conditions for the sentences of the language. Instead of accepting these limitations, §2.3 will review the semantics that Prior [16] and Thomason [17] provide for tensed languages. Although inadequate for interpreting bimodal languages, these accounts motivate the semantics developed in §3 in which possible worlds are defined in terms of *world states*, *tasks*, and *times* rather than being taken to be primitive.

---

<sup>8</sup>In the only example that Kripke [5] provides, the sentence ‘Sherlock Holmes is bald’ has a truth-value even though a time has not been specified and the name ‘Sherlock Homes’ does not refer. Assuming Cary to be in the flesh and blood, ‘Cary is reading’ may be said to have a truth-value with considerably less controversy. In both cases, Kripke’s semantics returns a truth-value despite the fact that a time has not been included in the sentence in question nor among the parameters at which it is interpreted.

<sup>9</sup>I will follow Montague [1] and Kaplan [2] in omitting consideration of the accessibility relation.

## 2.1 Necessarily Always

Whereas Montague [1] sought to provide a semantics for a fragment of English by translating that fragment into a quantified bimodal language for which he gave a recursive semantic theory, I will restrict consideration to the propositional fragment of Montague’s bimodal language. To facilitate comparison, I will include  $\boxtimes$  in place of  $\Box$  in the propositional language  $\mathcal{L}^M = \langle \mathbb{L}, \perp, \rightarrow, \boxtimes, \boxplus, \boxminus \rangle$ .<sup>10</sup> By defining the well-formed sentences of  $\mathcal{L}^M$  in the usual way, we may extend the interpretation provided by each two-dimensional model  $\mathcal{M}_2$  to all well-formed sentences of  $\mathcal{L}^M$  as follows:

- $(p_i)$   $\mathcal{M}_2, w, x \models p_i$  iff  $\langle w, x \rangle \in |p_i|$ .
- $(\perp)$   $\mathcal{M}_2, w, x \not\models \perp$ .
- $(\rightarrow)$   $\mathcal{M}_2, w, x \models \varphi \rightarrow \psi$  iff  $\mathcal{M}_2, w, x \not\models \varphi$  or  $\mathcal{M}_2, w, x \models \psi$ .
- $(\boxtimes)$   $\mathcal{M}_2, w, x \models \boxtimes\varphi$  iff  $\mathcal{M}_2, u, y \models \varphi$  for all  $u \in W$  and  $y \in T$ .
- $(\boxplus)$   $\mathcal{M}_2, w, x \models \boxplus\varphi$  iff  $\mathcal{M}_2, w, y \models \varphi$  for all  $y \in T$  where  $y < x$ .
- $(\boxminus)$   $\mathcal{M}_2, w, x \models \boxminus\varphi$  iff  $\mathcal{M}_2, w, y \models \varphi$  for all  $y \in T$  where  $x < y$ .

Postponing the controversies that beset the interpretation of the future tense operator to §2.3 and §4.1, I will take the semantics for both the extensional and tense operators to be uncontroversial for the time being.<sup>11</sup> Whereas Montague [1] took modal claims of the form ‘ $\boxtimes\varphi$ ’ to be read ‘It is necessarily the case that  $\varphi$ ’, the same paper in the edited collection included an editor’s note by Richard Thomason [18, p. 259, FN 9] that  $\boxtimes$ , “is interpreted in the sense of ‘necessarily always’.” After all,  $\boxtimes\varphi$  is true in a world  $w$  at a time  $x$  just in case  $\varphi$  is true in all worlds at all times, quantifying over both the modal and temporal dimensions of Montague’s semantics at once.

By letting a *moment* be any ordered pair  $\langle w, x \rangle \in W \times T$ , the semantics for  $\boxtimes$  quantifies over all moments. By contrast, I will take the following *universal semantics* for  $\Box$  to quantify over all worlds while leaving the temporal parameter unchanged:

- $(\Box)$   $\mathcal{M}, w, x \models \Box\varphi$  iff  $\mathcal{M}, u, x \models \varphi$  for all  $u \in W$ .<sup>12</sup>

To evaluate whether  $\Box$  or  $\boxtimes$  expresses metaphysical necessity, I will take  $\mathcal{L}$  be the result of replacing  $\boxtimes$  in  $\mathcal{L}^M$  with  $\Box$  while maintaining the other operators included in the language. Next we may derive the semantic clause for  $\Delta$  from its definition above:

- $(\Delta)$   $\mathcal{M}, w, x \models \Delta\varphi$  iff  $\mathcal{M}, w, y \models \varphi$  for all  $y \in T$ .

Extending  $\mathcal{L}$  to include  $\boxtimes$  as a primitive symbol makes  $\boxtimes\varphi \leftrightarrow \Box\Delta\varphi$  valid by being true in every world at every time on any two-dimensional model, thereby rendering  $\boxtimes$  redundant. Rather, I will maintain  $\boxtimes\varphi := \Box\Delta\varphi$  as a metalinguistic abbreviation in  $\mathcal{L}$ , deriving the semantics for  $\boxtimes$  from its definition. Since **T1** in §5 proves that  $\Box$

<sup>10</sup>I will assume  $\neg\varphi := \varphi \rightarrow \perp$ ,  $\varphi \vee \psi := \neg\varphi \rightarrow \psi$ ,  $\varphi \wedge \psi := \neg(\varphi \rightarrow \neg\psi)$ , and  $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ .

<sup>11</sup>Although the following discussion will concern the interactions between both tense operators and  $\boxtimes$ , these arguments could be restated in terms of just the past tense operator which is much less controversial.

<sup>12</sup>For the sake of the comparison below, quantification over worlds has not been restricted by accessibility.

cannot similarly be defined in  $\mathcal{L}^M$ , it follows that  $\mathcal{L}$  is more expressive than  $\mathcal{L}^M$ . For the purposes of comparing  $\Box$  and  $\boxtimes$  in a common language,  $\mathcal{L}$  enjoys a clear advantage over working in  $\mathcal{L}^M$ . Assuming that  $\mathcal{L}^M$  is to be replaced by  $\mathcal{L}$  where  $\boxtimes$  is then defined in terms of  $\Box$  and  $\Delta$  as above, the question remains whether or not to read ' $\Box\varphi$ ' as 'It is necessarily the case that  $\varphi$ ' in accordance with Thomason's suggested reading of ' $\boxtimes\varphi$ ' as 'It is necessarily always the case that  $\varphi$ '.

In opposition to Thomason's reading, Dorr and Goodman [15, p. 636] take  $\boxtimes$  and  $\Diamond\varphi := \neg\boxtimes\neg\varphi$  to be the metaphysical modals rather than  $\Box$  and  $\Diamond\varphi := \neg\Box\neg\varphi$ , appealing to the fact that **P1** is invalid over the class of two-dimensional models. So long as there are multiple times in  $T$ , we may consider a model where  $p_i$  is assigned to a set of world-time pairs that includes  $\langle w', x \rangle$  for all  $w' \in W$  at a fixed time  $x \in T$  but that does not include  $\langle w, x' \rangle$  for all times  $x' \in T$  at a fixed world  $w \in W$ . It follows that  $\Box p_i$  may be true in  $w$  at  $x$  while  $\Delta p_i$  is false, thereby invalidating **P1** where **P2** is equivalent. By contrast, replacing  $\Box$  and  $\Diamond$  in **P1** and **P2** with  $\boxtimes$  and  $\Diamond$  may be shown to be valid. More specifically, consider the following *trivial perpetuity principles*:

$$\mathbf{TP} \quad \boxtimes\varphi \rightarrow \Delta\varphi.$$

$$\mathbf{CT} \quad \nabla\varphi \rightarrow \Diamond\varphi.$$

Following Montague in assuming that  $\boxtimes$  and  $\Diamond$  are the metaphysical modals, Dorr and Goodman take **TP** and **CT** to express the perpetuity principles, citing their intuitive plausibility given a metaphysical reading of  $\boxtimes$  and  $\Diamond$ . However, instead of expressing substantive interaction principles for tense and modality, **TP** and **CT** are valid for entirely modal reasons. Given the definition  $\boxtimes\varphi := \Box\Delta\varphi$ , we may observe that **TP** is an instance of the **T** axiom  $\Box\psi \rightarrow \psi$  which is valid over the two-dimensional models of  $\mathcal{L}$  merely by quantifying over possibilities, where **CT** is similar. By contrast, I will assume that the validity of the perpetuity principles ought to follow from the interaction between the modal and temporal dimensions of the semantics.

Denying that  $\Box$  and  $\Diamond$  are the metaphysical modals does not answer the objection that the validity of **TP** follows too easily by being an instance of the **T** axiom, where **CT** is equivalent. Even if it is admitted that  $\boxtimes$  and  $\Diamond$  are the metaphysical modals as Dorr and Goodman insist, **TP** and **CT** are valid merely by virtue of the fact that  $\Box\varphi \rightarrow \varphi$  and  $\varphi \rightarrow \Diamond\varphi$  are valid, neither of which include temporal operators. To avoid appealing to the status of these principles in the proof system for one language over another, we may put the point purely semantically by way of the following:

$$\begin{aligned} \text{Triviality: } (\forall w \in W \forall x \in T : \mathcal{M}, w, x \models \varphi) &\rightarrow (u \in W \rightarrow \forall x \in T : \mathcal{M}, u, x \models \varphi). \\ (u \in W \wedge \exists x \in T : \mathcal{M}, u, x \models \varphi) &\rightarrow (\exists w \in W \exists x \in T : \mathcal{M}, w, x \models \varphi). \end{aligned}$$

The principles above are instances of the first-order theorems  $\forall w \Psi \rightarrow \Psi[u/w]$  and  $\Psi[u/w] \rightarrow \exists w \Psi$  by replacing  $\Psi$  with  $w \in W \rightarrow \forall x \in T : \mathcal{M}, w, x \models \varphi$  or else with  $w \in W \wedge \exists x \in T : \mathcal{M}, w, x \models \varphi$ , respectively. Although  $\Psi$  includes quantification over times, nothing in  $\Psi$  accounts for why the *Triviality* principles are valid. Rather, these principles are valid on account of universal instantiation and existential generalization with respect to quantification the over worlds in  $W$  independent of the quantification over times in  $\Psi$ . This outcome fails to satisfy the expectation that the perpetuity principles are valid by virtue of the interaction between worlds and times.

Whereas **TP** and **CT** are instances of  $\Box\varphi \rightarrow \varphi$  and  $\varphi \rightarrow \Diamond\varphi$  which describe the purely modal dimension of the semantics encoded by  $W$ , the perpetuity principles **P1** and **P2** constrain the interaction between the distinct dimensions  $T$  and  $W$  included in the semantics for tense and modality. By contrast, the trivial principles **TP** and **CT** do not express a significant relationship between  $T$  and  $W$ , undermining their interest as genuine interaction principles. However, instead of defending the meaningfulness of the possible worlds in  $W$  included in the semantics, Dorr and Goodman [15, p. 646], “think it best to avoid ‘world’-talk altogether in theorizing about temporary matters.” In opposition to this perspective, I will assume that the semantic primitives included in the models of a language ought to provide the intuitive bedrock by which to define meaningful truth-conditions for the well-formed sentences of the language.

Despite undermining the significance of  $\Box$  by rejecting the intended interpretation of  $W$  as the set of possible worlds, Dorr and Goodman draw on the resources that Fine [19] provides to define  $\Box$  partially in terms of  $\boxtimes$ . By including a countable set of *time variables*  $\mathcal{V} := \{t_i : i \in \mathbb{N}\}$  in the language which may be bound by first-order quantifiers and letting an *assignment* be any function  $g : \mathcal{V} \rightarrow T$  from time variables in  $\mathcal{V}$  to times in  $T$ , Dorr and Goodman [15, pp. 635, 655] provide the following semantic clauses in order to define  $\Box$  in terms of their preferred primitive  $\boxtimes$ :

( $\exists t$ )  $\mathcal{M}, w, x, g \models \exists t\varphi$  iff  $\mathcal{M}, w, x, g' \models \varphi$  for some  $g'$  differing from  $g$  at most in  $t$ .

(Pr)  $\mathcal{M}, w, x, g \models \text{Present}(t)$  iff  $g(t) = x$ .<sup>13</sup>

*Converse Definition:*  $\Box\varphi := \exists t[\text{Present}(t) \wedge \boxtimes(\text{Present}(t) \rightarrow \varphi)]$ .

By taking  $\mathcal{L}^F$  to extend  $\mathcal{L}^M$  to include the Finean resources given above, **L2** shows that  $\Box\varphi$  is true in world  $w$  at time  $x$  on assignment  $g$  just in case  $\varphi$  is true in world  $u$  at time  $x$  on assignment  $g$  for all worlds  $u \in W$  in perfect alignment with the universal semantic clause provided above for  $\Box$ . Nevertheless, **T2** in the *Appendix* shows that **P1** and **P2** are both invalid on account of the fact that  $\Box$  quantifies over  $W$  and  $\Delta$  quantifies over  $T$  where  $W$  and  $T$  are entirely independent.

Whereas  $\boxtimes$  is easy to define in terms of  $\Box$  and  $\Delta$  in  $\mathcal{L}$ , the definition that Dorr and Goodman provide for  $\Box$  in terms of  $\boxtimes$  and first-order quantification over times is far from natural. Rather, the ideological complexity of the *Converse Definition* reflects the relative obscurity of  $\Box$  from the perspective of the primitives included in  $\mathcal{L}^F$ . Dorr and Goodman [15] then proceed to make the following claim:

But even if we were convinced that there was a practice afoot of using ‘metaphysically necessary’ to express  $\Box$ , we would still emphatically reject the claim that this way of speaking was “just as good” as ours [i.e.,  $\boxtimes$ ]. For there are hypotheses about the possible structures of time that simply cannot be expressed in the language of tense operators, propositional quantifiers, and an operator  $\Box$ . (p. 656)

In support of these remarks, Dorr and Goodman present the following hypothesis:

<sup>13</sup>Dorr and Goodman [15, p. 637] admit that Fine’s [19] use of propositional quantifiers to eliminate time variables from the expanded language behaves pathologically when  $\Box$  is replaced by  $\boxtimes$ , concluding that, “for proponents of [TP], quantification over times provides a kind of co-ordination between different possible world-histories that cannot be expressed using only standard modal and temporal operators,” further undermining the interest of the operator  $\boxtimes$  that they take to be primitive.

**H1** It is metaphysically necessary that time is dense.

Although Dorr and Goodman claim that, “there seems to be no way of conducting such a debate using  $\Box$  as one’s basic modal notion” (p. 656), the opposite is true.<sup>14</sup> Rather, anything that can be expressed in  $\mathcal{L}^M$  can just as easily be articulated in  $\mathcal{L}$ . Even given the extra ideology of  $\mathcal{L}^F$ , it is cumbersome to make use of an operator which quantifies over the moments in  $W \times T$  rather than the possible worlds in  $W$  given that tense operators have already been included in the language. Moreover, §3 provides a semantics for  $\mathcal{L}$  where **P1** and **P2** are not only valid, but **H1** is valid over all models in which time is dense despite admitting countermodels when time is discrete.

In addition to its greater expressive power, the universal semantics for  $\mathcal{L}$  makes the perpetuity principles **P1** and **P2** substantive by taking the tense and modal operators to quantify over distinct dimensions of the model theory. By contrast, the trivial principles **TP** and **CT** are instances of the **T** axiom for  $\Box$  and so although valid, fail to capture the significance of **P1** and **P2**. Moreover, following Dorr and Goodman in attempting to overcome the expressive limitations of the Montagovian semantics for  $\Box$  by relying on first-order quantification over times not only threatens to make the temporal operators superfluous, but also posits a vast ontology of times. Rather than attempting to explain away appearances, I will set the Montagovian semantics aside. By excluding first-order quantification over times from the object language, I will restrict attention to the interaction between the tense and modal operators included in  $\mathcal{L}$ . In the following subsection, I will consider attempts to validate the perpetuity principles by constraining the class of models, arguing that the resulting theory is committed to an absolute theory of time that cannot be avoided without undermining the significance of the truth-conditions for the sentences of the language.

## 2.2 Absolute Time

Instead of following Montague [1] in taking the semantics for the metaphysical modals to quantify over the moments in  $W \times T$ , Kaplan’s [2] semantics for  $\Box$  and  $\Diamond$  quantifies over just the worlds  $W$  included in a two-dimensional model of the language  $\mathcal{L}$ . As **T2** shows, Kaplan’s semantics permits a sentence letter to be true at every world at a given time without also being true at every time in a given world, making the perpetuity principles invalid over the class of all two-dimensional models. Although Kaplan does not consider **P1** and **P2**, I show in **T3** that **P1** and **P2** are valid over the abundant two-dimensional models of  $\mathcal{L}$  defined below where **T4** shows that time is unbounded in abundant models with at least two distinct times:<sup>15</sup>

---

<sup>14</sup>Dorr and Goodman also consider the much more esoteric claim that, “Although it is only contingently true that time is dense, it is necessary that for each time  $t$ , either it is necessary that if  $t$  is ever present, time is dense, or else it is necessary that if  $t$  is ever present, time is not dense.” Since this is not a bimodal claim that expresses the interactions between tense and modal operators, it is outside the scope of this paper.

<sup>15</sup>A weak partial order  $\langle T, \leq \rangle$  is *bounded* just in case both of the following hold and *unbounded* otherwise:

*Bounded Below:* There is some  $y \in T$  where  $y \leq x$  for all  $x \in T$ .

*Bounded Above:* There is some  $y \in T$  where  $x \leq y$  for all  $x \in T$ .

*Time-Shift:* The worlds  $w, w' \in W$  are *time-shifted from  $y$  to  $x$* — i.e.,  $w \approx_y^x w'$ — in a model  $\mathcal{M}_2$  of  $\mathcal{L}$  iff there is an order automorphism  $\bar{a} : T \rightarrow T$  where  $y = \bar{a}(x)$  and the following holds for all sentence letters  $p_i \in \mathbb{L}$  and times  $z \in T$ :

$$\langle w', z \rangle \in |p_i| \Leftrightarrow \langle w, \bar{a}(z) \rangle \in |p_i|.^{16}$$

*Abundance:* A two-dimensional model  $\mathcal{M}_2$  of  $\mathcal{L}$  is *abundant* iff for every  $w \in W$  and  $x, y \in T$ , there is some  $w' \in W$  that is time-shifted from  $y$  to  $x$ , i.e.,  $w \approx_y^x w'$ .

Paradigm examples of abundant models identify  $T$  with either the set of integers  $\mathbb{Z}$ , rational numbers  $\mathbb{Q}$ , or real number  $\mathbb{R}$ . Nevertheless, it is natural to take some systems such as a game of chess to have a bounded set of times. That abundant models cannot accommodate bounded temporal orders is a significant limitation. Although one might appeal to the plausibility of **P1** and **P2** to defend a restriction to abundant models, taking an intended model to be abundant also vastly expands its primitive ontology. Letting a sentence  $\varphi$  be *temporary* in  $\mathcal{M}_2$  just in case  $\mathcal{M}_2, w, x \models \varphi$  and  $\mathcal{M}_2, w, y \not\models \varphi$  for some world  $w$  and times  $x$  and  $y$ , it follows that abundant models with temporary sentences include all merely temporal differences between worlds. For instance, given an abundant model  $\mathcal{M}_2$  in which  $\varphi$  is temporary, there is a world  $w$  and times  $x < y$  where  $\mathcal{M}_2, w, x \models \varphi$  and  $\mathcal{M}_2, w, y \not\models \varphi$ , and so by *Abundance* there is a world  $w'$  that is time-shifted  $w \approx_y^x w'$  from  $y$  to  $x$ . As shown in **L4**, what is true in  $w$  at  $y$  is exactly the same as what is true in  $w'$  at  $x$ . Since  $w \approx_y^x w'$ , it follows more generally that the same sentences are true in  $w$  at  $\bar{a}(z)$  and in  $w'$  at  $z$  for any time  $z$  whatsoever. Insofar as  $W$  represents a meaningful range of distinct possible worlds without ever representing the same history twice, it follows that abundant models with temporary sentences include all merely temporal differences between possible worlds.

I will refer to the thesis that there are merely temporal differences between possible worlds as *temporal absolutism*, a thesis that I will assume it is natural to resist. Despite validating the perpetuity principles, restricting to the abundant models gives rise to an unfortunate trade-off when interpreting the intended models of  $\mathcal{L}$  since either one must embrace absolutism or else admit an excess of possible worlds that represent the same possible history many times over. Although it is preferable to avoid admitting redundancies among the semantic primitives included in a model, one might attempt to defend a restriction to abundant models by taking the extra time-shifted worlds to be empty artifacts of abundant models. However, if sets of world-time pairs are to provide meaningful truth-conditions for the well-formed sentences of  $\mathcal{L}$ , the worlds in  $W$  cannot be entirely void of significance. An *abundance theorist* might then attempt to maintain intelligible truth-conditions by abstracting from the merely temporal differences that hold between the worlds in  $W$  by way of the following definitions:

---

<sup>16</sup>Intuitively, an order automorphism  $\bar{a} : T \rightarrow T$  shifts the times in  $T$  without changing their order. More precisely, an *order automorphism* on the structure  $\langle T, \leq \rangle$  is a function  $\bar{a} : T \rightarrow T$  which has the properties:

*Injective:* For all  $x, y \in T$ , if  $f(x) = f(y)$  then  $x = y$ .

*Surjective:* For all  $y \in T$  there exists some  $x \in T$  where  $f(x) = y$ .

*Monotonic:*  $f(x) \leq f(y)$  whenever  $x \leq y$ .

Matters are much more complicated were one to wish to accommodate bounded sets of times  $T$ . See §3 for a considerably simpler solution on behalf of the construction of possible worlds.

*Time-Shifted Worlds:*  $w \approx w'$  iff  $w \approx_y^x w'$  for some  $x, y \in T$ .

*World Abstraction:*  $[w] := \{w' \in W \mid w \approx w'\}$ .

*Possible Worlds:*  $W_\approx := \{[w] \subseteq W \mid w \in W\}$ .

By letting  $[w]$  be the set of time-shifted worlds that represent the same possible world as  $w$ , an abundance theorist might claim that  $W_\approx$  represents the range of genuinely distinct possible worlds rather than the primitive set  $W$ . Although an abundant model which admits temporary sentences is guaranteed to include merely temporal differences between the primitive worlds in  $W$ , the elements in  $W_\approx$  abstract from the merely temporal differences between worlds. An abundance theorist may take **T3** to show how to validate **P1** and **P2** by restricting consideration to the abundant models without embracing absolutism or undermining the interpretation of possible worlds.

Despite providing a method for identifying a meaningful range of genuinely distinct possible worlds for each abundant model of  $\mathcal{L}$ , it is important to observe that  $W_\approx$  is defined in terms of the interpretation of the sentence letters of the language. Even if the worlds  $w \approx w'$  are time-shifted in a model  $\mathcal{M}_2$  of  $\mathcal{L}$  and so  $[w] = [w']$ , it does not follow that  $w \approx w'$  in any other model  $\mathcal{M}'_2$  of  $\mathcal{L}$ . For instance, assuming  $w \approx_y^x w'$  in  $\mathcal{M}_2$  for just  $x, y \in T$  where  $x \neq y$  and  $\langle w, y \rangle \in |p_0|$ , there is an automorphism  $\bar{a} : T \rightarrow T$  where  $y = \bar{a}(x)$  and  $\langle w', x \rangle \in |p_0|$ . By letting  $\mathcal{M}'_2$  be identical to  $\mathcal{M}_2$  except for taking  $\langle w', x \rangle \notin |p_0|$ , it follows that  $w \not\approx_y^x w'$  in  $\mathcal{M}'_2$  for any  $x, y \in T$ , and so  $[w] \neq [w']$  in  $\mathcal{M}'_2$ . This construction demonstrates how the equivalence classes in  $W_\approx$  may expand or contract depending on the truth-conditions assigned to the sentence letters by a model  $\mathcal{M}_2$  of the language  $\mathcal{L}$ . As a result, an abundance theorist cannot appeal to  $W_\approx$  in order to specify meaningful truth-conditions for the sentence letters of  $\mathcal{L}$  without circularity. In particular, consider the following definitions:

*Time-Shifted Moments:*  $[w, x] := \{\langle u, y \rangle \mid w \approx_y^x u\}$ .

*Truth-Conditional Abstraction:*  $|p_i|_\approx := \{[w, x] \mid \langle w, x \rangle \in |p_i|\}$ .

The first definition identifies the genuinely distinct moments in  $\mathcal{M}_2$  by abstracting from the differences between moments which make the same sentences true in  $\mathcal{M}_2$ . However, for  $|p_i|_\approx$  to provide a meaningful truth-condition for  $p_i$  in  $\mathcal{M}_2$ , one must already have an independent grasp of the original truth-condition  $|p_i|$  provided by  $\mathcal{M}_2$ , undermining the need to specify  $|p_i|_\approx$  in the first place. In addition to circularity, taking the truth-conditions for the sentence letters of  $\mathcal{L}$  to be the basis upon which to identify the range of genuinely distinct possible worlds  $W_\approx$  puts the cart before the horse. Rather, intended models provide a range of meaningful semantic primitives by which to specify the truth-conditions for all sentences of the language.

It is worth comparing a similar strategy applied to the semantics of a first-order extensional language  $\mathcal{L}^1$ . Given a domain  $D$  that represents a meaningful range of primitive *objects*, a truth-conditional semantics may interpret  $\mathcal{L}^1$  by specifying the extensions of the constants and  $n$ -place predicates as elements of  $D$  and subsets of  $D^n$ . For instance, we may interpret a constant  $c$  in a model by assigning it to an element of  $D$  and interpret a one-place predicate  $F$  in a model by taking its extension to be a subset of the domain  $D$ . However, if it is denied that  $D$  includes genuinely distinct

objects but rather some other entities which may represent the same genuinely distinct object many times over, we lose our initial grasp on the meaning of the constant  $c$  and the predicate  $F$ . If  $D$  is left uninterpreted, or else interpreted in terms of another assignment of constants and predicates to extensions in  $D^n$ , then we cannot look to  $D$  to provide an independent basis upon which to interpret the language  $\mathcal{L}^1$ .

In keeping with a standard methodology in truth-conditional semantics, I will assume that the semantic primitives included in an intended model for a given object language provide an independently meaningful basis upon which to make sense of the truth-conditions for that language. Rather than presuming that the only way for the semantic primitives in an intended model to be meaningful is for those primitives to be identical to the parts of the reality that they model, I will take the intended models for a semantics to provide an idealization that simulates what the object language seeks to express with the expressive resources of a well-theorized metalanguage. I will refer to this approach as *simulation metasemantics* in opposition to both a *realist metasemantics* which takes the intended model to include the constituents of reality that the object language represents as well as to an *instrumentalist metasemantics* in which the semantic primitives have no significance beyond the instrumental role they play in the semantics. To validate **P1** and **P2** without restricting consideration to abundant models, §3 will present an alternative to the two-dimensional semantics given above while maintaining a traditional approach to truth-conditional semantics that is compatible with a simulation metasemantics. Rather than taking possible worlds to be primitive points devoid of structure, the semantics I will provide for  $\mathcal{L}$  constructs possible worlds from *world states*, *tasks*, and *times*. As I will show, the resulting model structures provide a natural first-order simulation of what the bimodal language  $\mathcal{L}$  is able to express. The following subsection motivates this approach by first considering Prior [16] and Thomason’s [17] semantic systems for tense logic.

## 2.3 World States

In *Time and Modality*, Prior [13] developed a Diodorean interpretation of  $\Diamond\varphi$  at a sequence of numbers which he took to represent the future.<sup>17</sup> Whereas the first number in the sequence represented the truth-value of  $\varphi$  in the present, the subsequent numbers represent the truth-value of  $\varphi$  at incrementally later times.

Commenting on Prior’s book, Kripke [20] observed in a letter that the Diodorean system validates  $\Box\Diamond\varphi \vee \Box\Diamond\neg\varphi$  which does not belong to an S4 logic. Kripke went on to suggest a branching structure in which the past is determined but the future remains open, where models of this kind are more appropriate to the S4 logic that Prior had discussed. Crediting Kripke for this account in [16, p. 27], Prior developed a number of semantic theories and corresponding logics for tensed languages where sentences are evaluated at *world states* which model, “instantaneous total states of the world,” (p. 88) rather than temporally extended histories. Intuitively, each world state is a complete configuration of the system under study at an instant. In order to

---

<sup>17</sup>Diodorus Cronus (died c. 284 BCE) was a Megarian logician who defined possibility and necessity in temporal terms, where a proposition is possible just in case it either is or will be true, and necessary just in case it is true and always will be true. Although Prior was concerned to provide a semantics and logic for a tensed language without metaphysical modal operators, he stops short of claiming that the metaphysical modals are definable in terms of the temporal operators with which he was concerned.

interpret the tense operators, Prior included a strict *earlier-than* relation  $<$  to order the world states. Although strict and weak orders are interdefinable in a first-order language with identity, I will follow Prior in taking  $<$  to be primitive for the time being. To situate his semantic approach in contrast to previous developments, Prior [16, p. 7-8] cites Broad's [21, p. 315] criticism of McTaggart's [22] famous argument that time is unreal, writing that the fundamental error in McTaggart's argument is his attempt to, "impose conditions appropriate to a tenseless language upon a tensed one." Prior [16, p. 12] avoids this defect by following Broad's suggestion to, "drop the temporal predicates 'past', 'present', and 'future'."<sup>18</sup> Whereas McTaggart includes times, temporal predicates, and tense operators all in one language in order to present his arguments, Prior interprets an object language with tense operators by appealing to an extensional metalanguage in which world states are ordered by an earlier-than relation but no other temporal relations or properties are included.

Although the world states form a strict total order in one of the simplest versions of his semantics, Prior also considered models which accommodate an open future by taking the world states to form a strict partial order that is connected and left-linear rather than total.<sup>19</sup> Despite these differences,  $<$  is a strict partial order on each account that Prior considered since no world state is permitted to be earlier than itself, and a world state  $s$  is earlier than  $r$  if both  $s$  is earlier than  $t$  and  $t$  is earlier than  $r$ . It is important to observe that taking world states to be strictly ordered prevents the same world state from occurring more than once in any history for that system. However, supposing that history never repeats itself is a substantive assumption which should not be built into the semantics. Moreover, there are systems which we may wish to study that admit loops in their evolution. For instance, taking the system in question to be a chessboard, there are histories for that system which include meandering end games where the same board state occurs more than once. As restricted as this system may be, nothing should rule out consideration of such games of chess. Rather, this example highlights a fundamental limitation of the theoretical roles that world states play in Prior's semantics. Although it is natural to consider systems which occupy the same instantaneous configuration at different times in that system's history, this is forbidden if world states are strictly ordered into a time series.

In order to disentangle the distinct theoretical roles which world states play in Prior's semantics, it will help to define a minimally constrained class of models that generalize on Thomason's [17] reconstruction of Prior's [16] Peircean and Ockhamist semantic theories. Letting  $\mathcal{L}^T := \langle \mathbb{L}, \perp, \rightarrow, \Box, \Diamond \rangle$  be the non-modal fragment of the language  $\mathcal{L}$ , a *strict model* of  $\mathcal{L}^T$  is an ordered triple  $\mathcal{P} = \langle T, <, |\cdot| \rangle$  where  $\langle T, < \rangle$  is a strict partial order for a nonempty set of world states  $T$ , and  $|p_i| \subseteq T$  for all sentence letters  $p_i \in \mathbb{L}$ . Although some strict models correspond to a single history for a system by also being total, strict models may in general accommodate distinct histories which

---

<sup>18</sup>Instead of following Reichenbach [23] in distinguishing between the point-of-speech, point-of-reference, and point-of-event to evaluate tensed sentences at a single world state, it is by evaluating sentences with nested tense operators that the evaluation time shifts between the right number of world states.

<sup>19</sup>A *strict partial order*  $\langle T, < \rangle$  is a set  $T$  equipped with an irreflexive and transitive relation  $<$ . Letting  $x \sim y := (x < y) \vee (x = y) \vee (x > y)$  express that  $x$  and  $y$  are comparable, a strict partial order  $\langle T, < \rangle$  is *total* if  $s \sim t$  for any  $s, t \in T$ . A strict partial order  $\langle T, < \rangle$  is *left-linear* just in case  $s \sim t$  for any  $s, t, r \in T$  where  $s < r$  and  $t < r$ . Letting  $\approx$  be the transitive closure of  $\sim$ , a frame  $\langle T, < \rangle$  is *connected* just in case  $x \approx y$  for all  $x, y \in T$ . A left-linear frame is connected if for any  $x, y \in T$ , there is some  $z$  where  $z < x$  and  $z < y$ .

do not intersect or only partially overlap. A *strict history* in a strict model  $\langle T, <, |\cdot| \rangle$  is any maximal total suborder  $h_i = \langle T_i, <_i \rangle$  of  $\langle T, < \rangle$ .<sup>20</sup> Letting  $H_{\mathcal{P}}$  be the set of all strict histories of a strict model  $\mathcal{P}$ , we may take  $H_{\mathcal{P}}^x := \{ \langle T_i, <_i \rangle \in H_{\mathcal{P}} \mid x \in T_i \}$  to be the strict histories  $\langle T_i, <_i \rangle$  of  $\mathcal{P}$  which include the world state  $x \in T_i$ .

Despite being defined rather than primitive, strict histories play a similar role to possible worlds in the semantics that Montague [1] and Kaplan [2] went on to provide since each strict history corresponds to a complete temporal evolution of the system in question. Given these definitions, we may contrast the following semantic clauses:

$$\begin{aligned} \text{Peircean: } \mathcal{P}, x \models \Box^{\text{P}} \varphi &\text{ iff } \mathcal{P}, y \models \varphi \text{ for some } h_i \in H_{\mathcal{P}}^x \text{ and all } y \in h_i \text{ where } y <_i x. \\ \mathcal{P}, x \models \Box^{\text{F}} \varphi &\text{ iff } \mathcal{P}, y \models \varphi \text{ for some } h_i \in H_{\mathcal{P}}^x \text{ and all } y \in h_i \text{ where } x <_i y. \\ \text{Ockhamist: } \mathcal{P}, h_i, x \models \Box^{\text{O}} \varphi &\text{ iff } \mathcal{P}, h_i, y \models \varphi \text{ for all } y \in Y \text{ where } y <_i x. \\ \mathcal{P}, h_i, x \models \Box^{\text{F}} \varphi &\text{ iff } \mathcal{P}, h_i, y \models \varphi \text{ for all } y \in Y \text{ where } x <_i y. \end{aligned}$$

Whereas Prior [16, p. 126, 132] provided a semantics for the metric tense operators and Thomason [17] restricted attention to  $\Diamond^{\text{P}}$  and  $\Diamond^{\text{F}}$ , I have derived the semantics for  $\Box^{\text{P}}$  and  $\Box^{\text{F}}$  from the semantics that Thomason provides while generalizing the account to accommodate all strict models of  $\mathcal{L}^{\text{T}}$ .<sup>21</sup> The Peircean and Ockhamist semantics disagree about what the tense operators express at any world state that belongs to more than one strict history. Whereas the Peircean semantics quantifies over all past or future world states in the strict history at which the tensed sentence is evaluated and that includes the world state, the Ockhamist semantics evaluates tensed sentences at both a strict history and world state in that history, quantifying over just the world states in that strict history. As a result, the Peircean semantics has a number of unnatural consequences. For instance, given a board state in a game of chess where there is at least one history in which Black does not make any further blunders, the sentence ‘Black is not going to blunder’ is true even though there may be other histories in which Black goes on to blunder. This is far from natural. Moreover, revising the Peircean semantics to quantify over *all* strict histories in addition to all past or future world states makes the operators  $\Box^{\text{P}}$  and  $\Box^{\text{F}}$  too strong and their duals too weak. For instance, in a game of chess in which there is at least one future in which the white king is in checkmate and at least one future in which the black king is in checkmate, both ‘Black is going to win’ and ‘White is going to win’ come out true.

Rather than quantifying over all past or future world states in either some or all strict histories, the Ockhamist semantics evaluates sentences at both a strict history and world state. Thus the sentences ‘Black is going to win’ and ‘White is going to win’ cannot both be true since there is no way for both kings to be checkmated in the same game. In addition to claiming this advantage, another prominent difference between the Ockhamist and Peircean semantics is the expressive power that they each afford. In a similar manner to the way in which the Montagovian semantics for  $\Box$  quantified over worlds and times at once, the Peircean semantics for the tense operators quantifies

<sup>20</sup>Letting a *total suborder* of  $\langle T, < \rangle$  be any total order  $\langle T_i, <_i \rangle$  where  $T_i \subseteq T$  and  $<_i$  is the restriction of  $<$  to  $T_i$ , a total suborder  $\langle T_i, <_i \rangle$  of  $\langle T, < \rangle$  is *maximal* just in case for any total suborder  $\langle T_j, <_j \rangle$  of  $\langle T, < \rangle$ , if  $\langle T_i, <_i \rangle$  is a total suborder of  $\langle T_j, <_j \rangle$ , then  $\langle T_j, <_j \rangle = \langle T_i, <_i \rangle$ .

<sup>21</sup>The Peircean semantics given above can be simplified were one to restrict consideration to strict models which are left-linear (similarly right-linear), since world states would then have a unique past (future).

over both strict histories and world states in the same semantic clauses. Since the Ockhamist tense operators only quantify over the world states in a given strict history, the Peircean operators may be defined in terms of the Ockhamist operators for tense given the *stability operator* which quantifies over intersecting strict histories:

$$(\Box) \quad \mathcal{P}, h_i, x \models \Box^o \varphi \text{ iff } \mathcal{P}, h_j, x \models \varphi \text{ for all } h_j \in H_{\mathcal{P}}^x.$$

Prior [16, p. 125] takes ' $\Box^o \varphi$ ' to read 'It is now unpreventable that  $\varphi$ ', though admits to using 'Necessarily  $\varphi$ ' for convenience, claiming that whereas the Peircean's, "rather strong 'will be' is simply the Ockhamist 'necessarily will be', the Ockhamist 'will be' [is] untranslatable," (p. 130) for the Peircean.<sup>22</sup> As Prior observes,  $\varphi \rightarrow \Box^o \varphi$  is valid for non-temporal sentences, where  $\Box$  (similarly  $\Box^o$ ) may be included in  $\varphi$  by restricting to strict models that are left-linear (right-linear). Although the Ockhamist may define the Peircean tense operators  $\Box^p \varphi := \Diamond^o \Box^o \varphi$  and  $\Box^p \varphi := \Diamond^o \Box^o \varphi$ , the Peircean is not in a position to define the Ockhamist operators for tense, nor to provide a semantics for a stability operator since sentences are evaluated at world states alone.<sup>23</sup>

Despite their differences, neither the Peircean nor Ockhamist semantics permit the same world state in  $T$  to occur more than once in a strict history, nor are there strict histories in which the same world states in  $T$  occur in different orders. Rather,  $\langle T, < \rangle$  is required to be a strict partial order for any strict model  $\langle T, <, |\cdot| \rangle$  of  $\mathcal{L}^T$ , and so if  $s <_i t$  for any strict history of  $\langle T, <, |\cdot| \rangle$ , then  $t \not\prec_j s$  for all strict histories of  $\langle T, <, |\cdot| \rangle$ . However, given Prior's conception of world states as complete configurations, there are systems in which the same world states occur more than once or in different orders in different possible histories. As brought out above, a chess game may include the same board state more than once, or two chess games may agree in all respects with the exception of a transposition of board states which occur in a different order.<sup>24</sup>

Insofar as  $T$  is taken to be the set of all world states of a particular system under study, the strict histories for that system do not permit the same world states to occur more than once or in different orders, significantly compromising the range of applications for the semantics. Rather than accepting these limitations, it is natural to reject Prior's conception of  $T$  as the set of world states where these are taken to be complete configurations of the system. Instead, I will take  $T$  to be the set of *times* in accordance with their ordering. After all,  $T$  is intended to provide resources for articulating a semantics for temporal operators which quantify over what comes before or after each element in  $T$ . By taking  $T$  to be the set of times, the truth-conditions for the sentences of  $\mathcal{L}^T$  specify *when* a sentence is true without indicating which world state the system occupies at each time. Given this reading of  $T$  as the set of times, strict models may include strict histories in which the same world states occur more than once or in different orders for the simple reason that strict models

<sup>22</sup>Prior [16, p. 130] attributes these observations to J.M. Shorter in 1957 but does not provide a proof.

<sup>23</sup>Attempting to overcome these expressive limitations by including a strict history among the evaluation parameters and adapting the Finean resources from §2.1 to define the Ockhamist operators with the Peircean operators abandons the spirit of Prior's semantics. Rather, Prior sought to avoid McTaggart's paradox by separating the tense operators in the object language  $\mathcal{L}^T$  from the world states  $T$  and earlier-than relation  $<$  provided by the models of  $\mathcal{L}^T$ . In order to avoid ontological commitment to the world states in  $T$ , I will follow Prior in restricting quantification over  $T$  to the metalanguage used to provide the semantics for  $\mathcal{L}^T$ .

<sup>24</sup>Following Muller's [24] suggestion, Rumberg [25] extends Prior [16] and Thomason's [17] semantics. By virtue of working over a partial order, this extension faces the same difficulties brought out here.

only concern times and do not say anything at all about world states. As a result, there is no telling which times in  $T$  occupy the same world states, or which strict histories include the same world states in different orders. For instance, supposing the white king to be in check at multiple times in a chess game, there is no way to distinguish which times in the truth-condition for ‘The white king is in check’ correspond to the same complete configuration of the chessboard (if any) and which times correspond to distinct configurations. Moreover, given two strict histories for a chess game which agree in all respects with the exception of a short sequence of moves in which the same board states occur in different orders, the divergent sequences must include distinct times in  $T$  despite reordering the same configurations of the chessboard.

Taking  $T$  to be the set of times rather than world states avoids ruling out strict histories in which the same world states occur more than once or in different orders. Since all that is required to interpret tense operators is to specify *when* each sentence is true, the erroneous interpretation of  $T$  as a set of complete configurations of a system may be forgiven for a non-modal language like  $\mathcal{L}^T$ . Despite providing an adequate range of semantic primitives for interpreting the tensed languages with which Prior and Thomason were concerned, the same cannot be said for a bimodal language with operators for both tense and metaphysical modality. In order to evaluate a sentence  $\varphi$  of the bimodal language  $\mathcal{L}$  at a model  $\mathcal{P}$ , strict history  $h_i$ , and time  $x$ , it is important that  $x \in T_i$  be a valid time in the strict history  $h_i = \langle T_i, <_i \rangle$ . As a result, the strongest modal operator that an Ockhamist model theory can interpret only quantifies over those strict histories in which the time of evaluation occurs, and so cannot quantify over the full range of strict histories given by a strict model. This is precisely what the Ockhamist stability modal  $\Box$  achieves, though Prior is careful not to conflate this modality with metaphysical necessity since the stability modal does not quantify over all strict histories whatsoever. Since there is no Ockhamist modality that is able to quantify over all strict histories, the range of strict histories cannot represent the range of all possible evolutions of the system in question.

Instead of adapting Prior’s semantics for tensed languages to bimodal languages by replacing strict histories with primitive worlds as in Montague [1] and Kaplan’s [2] accounts, the following section will take both *times* and *world states* to be primitive, clearly distinguishing the theoretical roles that Prior conflates when interpreting his semantics. By taking sets of world states to provide truth-conditions for the sentences of the language, possible worlds will be defined as appropriately constrained functions from times to world states which trace out different paths through the space of all world states. In addition to providing an intuitive model theory in which possible worlds may occupy the same world states more than once or in different orders, **T5** in the *Appendix* shows that **P1** and **P2** are valid over an unrestricted class of models.

### 3 Possible Worlds

In order to interpret the bimodal language  $\mathcal{L}$ , sentences will be evaluated at both a world history and time in addition to a model of  $\mathcal{L}$ . Rather than following Montague [1] and Kaplan [2] by taking possible worlds to be primitive, I will define world histories

to be functions from times to world states.<sup>25</sup> Whereas the world states are interpreted as the instantaneous maximal possible configurations of the system under study, the times are mere indexes by which to parameterize the possible trajectories through the space of world states. Accordingly, I will take the times and not the world states to form a total order while also positing additive group structure so that times can be added and subtracted. The models of  $\mathcal{L}$  will include an interpretation function that maps each sentence letter to a set of world states which provides the truth-condition for that sentence. Since each set of world states represents the ways for the system to be in which a given sentence is true without including any temporal elements, times are entirely exogenous to the truth-conditions for the sentence letters.

Positing a set of world states  $W$  in addition to a set of times  $T$  distinguishes the theoretical roles which Prior [16] and Thomason [17] conflate. The truth-condition for a sentence of the bimodal language  $\mathcal{L}$  specifies all the configurations in which that sentence is true, not just the times at which it happens to be true. In order to add and subtract times without positing any additional structure, I will take a *temporal order* to be any totally ordered abelian group  $\mathcal{T} = \langle T, +, \leq \rangle$ .<sup>26</sup> Instead of admitting the incomparable times permitted by a partial order,  $\mathcal{T}$  will be used to identify all possible totally ordered world histories where world histories may diverge in the world states that they occupy at different times. Since not every path through the space of world states counts as a possible world history, I will equip frames with a *parameterized task relation*  $w \Rightarrow_x u$  to encode which transitions between world states  $w, u \in W$  with a duration  $x \in T$  are possible for a given system.<sup>27</sup> The task relation abstracts from the universal laws that one might hope to articulate for that system in order to describe and predict which transitions are possible. For instance, specifying the rules of chess determines the extension of the task relation where any game of chess must begin with the initial board state and all transitions between board states conform to the rules. More generally, I will define a *frame* to be any  $\mathcal{F} = \langle W, \mathcal{T}, \Rightarrow \rangle$  where  $W$  is a nonempty set of *world states*,  $\mathcal{T}$  is a temporal order, and  $\Rightarrow$  satisfies the constraints:

*Nullity:*  $w \Rightarrow_0 w$ .

*Compositionality:* If  $w \Rightarrow_x u$  and  $u \Rightarrow_y v$ , then  $w \Rightarrow_{x+y} v$ .

Each frame provides what is also called a *non-deterministic dynamical system* from dynamical systems theory.<sup>28</sup> Postponing further consideration of this connection to §4.2, I will define a *world history* to be a function  $\tau : X \rightarrow W$  where  $X \subseteq T$  is convex and  $\tau(x) \Rightarrow_y \tau(x+y)$  for all times  $x, y \in T$  where both  $x, x+y \in X$ . Although the particular times in a world history may have different order types in that history, the

---

<sup>25</sup> Although the world states will be primitive for our purposes, I define world states in [26] by appealing to the task and parthood relations defined over a broader space of states.

<sup>26</sup> A *group* is any  $\langle G, \cdot \rangle$  where: (1)  $a \cdot b \in G$  whenever  $a, b \in G$ ; (2)  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  for all  $a, b, c \in G$ ; (3)  $1_G \in G$  where  $a \cdot 1_G = 1_G \cdot a = a$  for all  $a \in G$ ; and (4) for each  $a \in G$ , there is some  $-a \in G$  where  $a \cdot (-a) = 1_G$ , written  $a - a = 1_G$  for ease. A group is *abelian* just in case  $a \cdot b = b \cdot a$  for all  $a, b \in G$ . A group is *totally ordered* by  $\leq$  just in case  $\leq$  is a total order where for all  $a, b, c \in G$ , if  $a \leq b$ , then  $a \cdot c \leq b \cdot c$ .

<sup>27</sup> Alternatively, one could introduce a *stochastic task function*  $\Phi_x(w, u) = p$  where  $p \in [0, 1]$  is a probability with either an objective or epistemic reading depending on the application.

<sup>28</sup> A more common notation with a sparser theory of *positive durations* in place of a temporal order takes a non-deterministic dynamical system to be any  $\mathbb{D} = \langle X, \mathcal{T}, \{R_t\}_{t \in \mathcal{T}} \rangle$  where  $\mathcal{T} = \langle T, + \rangle$  is a monoid and  $R_t \subseteq X \times X$  satisfies  $R_0 = 1_X$  and  $R_{s+t} = R_s \circ R_t$  for all  $s, t \in T$ .

times in  $T$  all have the same order type. For instance, although the times in a particular game of chess may be told apart by their distance from the initial board state, nothing turns on the particular sequence of times used to parameterize the board states in that game of chess. Instead of taking 0 to indicate the initial board state, one could just as well begin counting from 1, or any other integer.

Although the times in  $T$  form a total order so that no time in any world history occurs more than once, world histories may assign multiple times to the same world state. By permitting world states to occur at more than one time in a world history, there may not be a single answer to the question which world states came before or after another world state in a given world history. Rather, it is by specifying a time that each world history determines which world states are in the past or future relative to that time. In the case of a meandering end game in chess, there is nothing to prevent the chessboard from occupying the same board state at more than one time.

Instead of positing an abundance of primitive time-shifted possible worlds, the same semantic primitives included in  $\mathcal{F}$  generate an abundance of world histories. Letting  $H_{\mathcal{F}}$  be the set of all world histories defined over the frame  $\mathcal{F}$ , we may take the semantic clauses for the modal operators to quantify over all world histories in  $H_{\mathcal{F}}$ . Certain applications may restrict consideration to the *complete world histories*  $H_{\mathcal{F}}^* := \{\tau \in H_{\mathcal{F}} \mid \text{dom}(\tau) = T\}$  which assign all times in  $T$  to world states in  $W$  or, alternatively, to the *length  $n$  world histories*  $H_{\mathcal{F}}^n := \{\tau \in H_{\mathcal{F}} \mid |\text{dom}(\tau)| = n\}$ . Although there are many restricted sets of world histories that one might consider, the metaphysical modals concern the broadest set  $H_{\mathcal{F}}$ . Letting  $\tau \approx_y^x \sigma$  indicate there is an order automorphism  $\bar{a} : T \rightarrow T$  which time-shifts  $\tau$  to  $\sigma$  so that  $\text{dom}(\sigma) = \bar{a}(\text{dom}(\tau))$ ,  $y = \bar{a}(x)$ , and  $\bar{a}(u) \leq \bar{a}(v)$  whenever  $u \leq v$ , we may let  $\tau \approx \sigma$  express that  $\tau \approx_y^x \sigma$  for some  $x, y \in T$ . Since  $\mathcal{T}$  is translation invariant, each time  $x \in T$  induces an automorphism  $\bar{a}(z) = z + x$  which shifts the temporal order forwards by  $x$ . We may then take the set  $[\tau]_{\mathcal{F}} := \{\sigma \in H_{\mathcal{F}} \mid \tau \approx \sigma\}$  of world histories time-shifted from  $\tau$  to represent a *possible world* where  $\mathbb{W}_{\mathcal{F}} := \{[\tau]_{\mathcal{F}} \subseteq H_{\mathcal{F}} \mid \tau \in H_{\mathcal{F}}\}$  is the set of all possible worlds defined over the frame  $\mathcal{F}$  for a given system. Whereas possible worlds correspond to distinct paths through the space of all world states  $W$ , world histories parameterize those paths by fixing an assignment of times to world states while acknowledging that the choice of times does not represent anything significant. Although the set of possible worlds may claim to hold a metaphysical standing that the set of world histories cannot, it is the world histories that will play an important role in the semantics. Since possible worlds will play no further role throughout what follows, I will refer to world histories as possible worlds for familiarity.

Having presented the construction of possible worlds, we may define the *models* of  $\mathcal{L}$  to be any tuple  $\mathcal{M} = \langle W, \mathcal{T}, \Rightarrow, |\cdot| \rangle$  where  $\mathcal{F} = \langle W, \mathcal{T}, \Rightarrow \rangle$  is a frame and  $|p_i| \subseteq W$  for every sentence letter  $p_i \in \mathbb{L}$ . The well-formed sentences of  $\mathcal{L}$  are evaluated at a possible world  $\tau \in H_{\mathcal{F}}$  and time  $x \in T$  in a model  $\mathcal{M}$  as follows:

- $(p_i) \mathcal{M}, \tau, x \models p_i$  iff  $x \in \text{dom}(\tau)$  and  $\tau(x) \in |p_i|$ .
- $(\perp) \mathcal{M}, \tau, x \not\models \perp$ .
- $(\rightarrow) \mathcal{M}, \tau, x \models \varphi \rightarrow \psi$  iff  $\mathcal{M}, \tau, x \not\models \varphi$  or  $\mathcal{M}, \tau, x \models \psi$ .

- ( $\Box$ )  $\mathcal{M}, \tau, x \models \Box\varphi$  iff  $\mathcal{M}, \sigma, x \models \varphi$  for all  $\sigma \in H_{\mathcal{F}}$ .
- ( $\Box$ )  $\mathcal{M}, \tau, x \models \Box\varphi$  iff  $\mathcal{M}, \tau, y \models \varphi$  for all  $y \in T$  where  $y < x$ .
- ( $\Box$ )  $\mathcal{M}, \tau, x \models \Box\varphi$  iff  $\mathcal{M}, \tau, y \models \varphi$  for all  $y \in T$  where  $x < y$ .

Despite taking possible worlds to be defined rather than primitive points, possible worlds and times play the same conceptual roles that they have traditionally played in bimodal frameworks. Given a possible world together with a time in a model of  $\mathcal{L}$ , we may determine the truth-value of any sentence in the language. Whereas the semantics for the metaphysical modals quantify over all possible worlds in  $H_{\mathcal{F}}$ , the semantics for the tense operators quantify over all times in  $T$ . Certain applications may restrict the tense operators to the domain  $\text{dom}(\tau)$  for the possible world  $\tau$  of evaluation or introduce semantic clauses for modal operators that quantify over the complete worlds in  $H_{\mathcal{F}}^*$  to avoid discrepancies between the times in each world. For other systems, temporal discrepancies between possible worlds are perfectly appropriate.

In order to get a better sense of the semantics, consider a particular chess game  $\alpha$  in which Black nearly checkmates White on move 31 before blundering the dark squared bishop. Playing on until move 47 in  $\alpha$ , Black manages to win the end game, saying:

(K) If I hadn't blundered my bishop, I would have won much sooner.

Although the present framework is not equipped to interpret tensed counterfactual conditionals, it is clear that Black is lamenting the existence of another possible game in which the blunder had been avoided.<sup>29</sup> Nevertheless, we may evaluate the following:

(P) Black could have checkmated White much sooner.

Given a sufficiently strong reading of 'could', we may regiment P in  $\mathcal{L}$  as  $\Diamond\Diamond p_w$  where  $p_w$  reads 'White is in checkmate'. The sentence  $\Diamond\Diamond p_w$  is true in  $\alpha$  at move 47 just in case there is a game  $\beta$  where  $\Diamond p_w$  is true at move 47. Although move 47 could not have been played in a game in which  $\Diamond p_w$  is true at move 47, the present framework nevertheless permits sentences to be evaluated at move 47 in  $\beta$ . For instance, if  $p_w$  is true at move 31 in  $\beta$ , then  $\Diamond p_w$  is true in  $\beta$  at move 47. By contrast, every sentence letter is false in  $\beta$  at move 47 given that  $\beta$  is not defined at move 47.<sup>30</sup>

Having provided a theory of truth for the language  $\mathcal{L}$ , it remains to provide a theory of logical consequence by which to survey the valid forms of reasoning warranted by the semantics. I will take the definition to assume the following standard form:

*Logical Consequence:*  $\Gamma \models \varphi$  iff for any model  $\mathcal{M}$  of  $\mathcal{L}$ , possible world  $\tau \in H_{\mathcal{F}}$ , and time  $x \in T$ , if  $\mathcal{M}, \tau, x \models \gamma$  for all  $\gamma \in \Gamma$ , then  $\mathcal{M}, \tau, x \models \varphi$ .

A well-formed sentence  $\varphi$  of  $\mathcal{L}$  is *valid* just in case  $\emptyset \models \varphi$ , dropping set notation for convenience. In addition to providing an intuitive and general framework for semantic

<sup>29</sup>I develop a hyperintensional semantics for tensed counterfactual conditionals in [26].

<sup>30</sup>Although one could restrict quantification to  $\text{dom}(\tau)$  and permute the operators to avoid quantifying outside  $\text{dom}(\tau)$ , P21 proves that these operators commute in this strongest version of the semantics.

theorizing, the present account validates a simple and strong logic for  $\mathcal{L}$  without imposing any frame constraints. As **T5** shows, the perpetuity principles are valid:

$$\mathbf{P1} \quad \Box\varphi \rightarrow \Delta\varphi.$$

$$\mathbf{P2} \quad \nabla\varphi \rightarrow \Diamond\varphi.$$

Suppose for contradiction that **P1** has a counterinstance, and so for some well-formed sentence  $\varphi$ , it is metaphysically necessary that  $\varphi$  and yet it is sometimes not the case that  $\varphi$ . More precisely,  $\mathcal{M}, \tau, x \models \Box\varphi$  and  $\mathcal{M}, \tau, x \not\models \Delta\varphi$ , and so  $\mathcal{M}, \tau, y \not\models \varphi$  for some  $y \in T$ . We may then define the possible world  $\sigma(z) = \tau(z - x + y)$  so that  $\mathcal{M}, \sigma, x \not\models \varphi$ . Thus it follows that  $\mathcal{M}, \tau, x \not\models \Box\varphi$ , contradicting the above. This proves that **P1** is valid where it follows by classical reasoning that **P2** is equivalent.

Instead of admitting countermodels to **P1** and **P2** as in Kaplan's [2] semantics, the present approach validates the perpetuity principles without imposing *ad hoc* model constraints that undermine the significance of the truth-conditions for the language. Not only do all paradigm examples conform to these principles, **P1** accords with a natural account of metaphysical modality as the strongest objective modality. Insofar as 'metaphysically necessary' expresses the strongest objective modality, we may insist that  $\varphi$  is not metaphysically necessary if  $\varphi$  ever fails to be the case. By defining possible worlds in terms of the world states, tasks, and times included in a frame, the present approach maintains the standard semantic clauses for both tense and modal operators. In addition to these merits, taking the metaphysical modals to quantify over possible worlds rather than world-time pairs provides a more natural and more expressive theory than Montague's [1] semantics. I will provide further abductive support for the task semantic given above by presenting a logic for  $\mathcal{L}$  in §3.2. Additionally, the following subsection will extend  $\mathcal{L}$  to include a stability operator in order to demonstrate the power of the present approach in contrast to two-dimensional semantic theories.

### 3.1 Restricted Modalities

The semantics for metaphysical necessity  $\Box$  quantifies over all possible worlds in  $H_{\mathcal{F}}$  without restriction, thereby validating an S5 modal logic. Although this is in keeping with the interpretation of metaphysical modality as the strongest objective modality, there are applications which call for restricted modal operators. For instance, given a world  $\tau \in H_{\mathcal{F}}$  and time  $x \in T$ , we may let  $\langle \tau \rangle_x := \{\sigma \in H_{\mathcal{F}} \mid \sigma(x) = \tau(x)\}$  be the set of possible worlds that intersect  $\tau$  at  $x$  in order to provide the following semantics:

$$(\Box) \quad \mathcal{M}, \tau, x \models \Box\varphi \text{ iff } \mathcal{M}, \sigma, x \models \varphi \text{ for all } \sigma \in \langle \tau \rangle_x.$$

A sentence  $\Box\varphi$  is true in a world  $\tau$  at a time  $x$  just in case  $\varphi$  is true at time  $x$  in every possible world that occupies the same world state as  $\tau$  at  $x$ . Letting  $\Diamond\varphi := \neg\Box\neg\varphi$ , the *stability operators*  $\Box$  and  $\Diamond$  may be used to define the following modals:

$$\textit{Will Always: } \Box_I\varphi := \Box\Box\varphi.$$

$$\textit{Could Always: } \Box_?\varphi := \Diamond\Box\varphi.$$

$$\textit{Will Eventually: } \Diamond_I\varphi := \Box\Diamond\varphi.$$

$$\textit{Could Eventually: } \Diamond_?\varphi := \Diamond\Diamond\varphi.$$

Given move number  $x$  in a chess game  $\alpha$ , the operators above may be used to quantify over the games of chess which occupy the same board state as the game  $\alpha$  at time  $x$ . Letting  $p_w$  be the sentence ‘The white king is in checkmate’, a game of chess may be resigned if White’s king will eventually be in checkmate:  $\Diamond p_w$ . By contrast, letting  $p_b$  be the sentence ‘The black king is in checkmate’, a game of chess is still worth playing if both Black and White could eventually win:  $\Diamond p_b \wedge \Diamond p_w$ .

Another natural restriction on the set of possible worlds arises from considering only those possible worlds which overlap with a given world up to the present time while possibly diverging in the future, and similarly for the past. More precisely:

*Open Futures:*  $|\tau\rangle_x := \{\sigma \in H_{\mathcal{F}} \mid \sigma(y) = \tau(y) \text{ for all } y \leq x\}$ .

*Open Pasts:*  $\langle\tau|_x := \{\sigma \in H_{\mathcal{F}} \mid \sigma(y) = \tau(y) \text{ for all } y \geq x\}$ .

Whereas  $|\tau\rangle_x$  is the set of all possible worlds that occupy the same world state as  $\tau$  at each time up to and including  $x$  while possibly diverging at later times,  $\langle\tau|_x$  is the set of all possible worlds that occupy the same world states as  $\tau$  at  $x$  and all later times while possibly diverging at earlier times. Given these definitions, we may introduce operators which quantify over these restricted sets of possible worlds:

$(\Box) \mathcal{M}, \tau, x \models \Box\varphi$  iff  $\mathcal{M}, \sigma, x \models \varphi$  for all  $\sigma \in |\tau\rangle_x$ .

$(\Boxleftarrow) \mathcal{M}, \tau, x \models \Boxleftarrow\varphi$  iff  $\mathcal{M}, \sigma, x \models \varphi$  for all  $\sigma \in \langle\tau|_x$ .

The *open future* operator  $\Box$  quantifies over all possible worlds that agree with the world of evaluation up to the time of evaluation. Since we are typically ignorant of the future, there is much greater occasion to contemplate the range of open futures at a moment than there is to quantify over the open pasts. Nevertheless, the *open past* operator  $\Boxleftarrow$  is intelligible and has been included for the sake of comparison.

Given that the sets of possible worlds  $\langle\tau\rangle_x$ ,  $|\tau\rangle_x$ , and  $\langle\tau|_x$  are definable in terms of the construction of possible worlds, there is no need to posit additional primitive accessibility relations between possible worlds in order to provide a semantics for the intersection, open future, and open past operators. By contrast, taking possible worlds to be structureless points requires each frame to include primitive accessibility relations  $R_{\times}$ ,  $R_{\triangleright}$ , and  $R_{\triangleleft}$  in order to identify the appropriate subsets of possible worlds for these operators to quantify over. Since not all accessibility relations provide appropriate restrictions on the space of possible worlds, a number of further frame constraints would have to be imposed if  $\Box$ ,  $\Boxleftarrow$ , and  $\Boxleftarrow$  are to maintain their intended readings. At the very least, we ought to expect  $R_{\triangleright}$  and  $R_{\triangleleft}$  to be restrictions of  $R_{\times}$  where the intersection of  $R_{\triangleright}$  and  $R_{\triangleleft}$  is nonempty. Even so, merely imposing a range of constraints on the accessibility relations in accordance with the intended readings of the operators does not uniquely determine their extensions. Although permissible, these theoretical costs are avoided entirely by the present theory. Instead of positing a range of frame constraints, the construction of possible worlds makes it provable that  $|\tau\rangle_x \subseteq \langle\tau\rangle_x$  and  $\langle\tau|_x \subseteq \langle\tau\rangle_x$  where  $|\tau\rangle_x \cap \langle\tau|_x = \langle\tau\rangle_x$  and  $\tau \in \langle\tau\rangle_x$ . Rather than positing primitive accessibility relations together with a range of constraints, we may define  $R_{\times}(\tau, \sigma) := \sigma \in \langle\tau\rangle_x$ ,  $R_{\triangleright}(\tau, \sigma) := \sigma \in |\tau\rangle_x$ , and  $R_{\triangleleft}(\tau, \sigma) := \sigma \in \langle\tau|_x$ , avoiding

the need to impose frame constraints which align with an intended interpretation. Since  $R_\times$ ,  $R_\triangleright$ , and  $R_\triangleleft$  are definable, we may omit the extra notation.

Rather than accepting the costs of imposing constraints on a primitive accessibility relation between possible worlds, a natural move by the lights of the present framework is to take the *task relation* to be four-place so that  $u \Rightarrow_x^w v$  indicates that it is possible for the world state  $u$  to transition to the world state  $v$  in duration  $x$ , all from the perspective of the world state  $w$ . Letting  $H_{\mathcal{F}}^w$  be the set of world histories defined over  $\Rightarrow^w$ , we may provide a semantic clause for the restricted modal operator  $\Box$ :

$$(\Box) \quad \mathcal{M}, \tau, x \models \Box\varphi \text{ iff } \mathcal{M}, \sigma, x \models \varphi \text{ for all } \sigma \in H_{\mathcal{F}}^{\tau(x)}.$$

Without assuming  $\Rightarrow^w$  to be invariant with respect to the world states  $w \in W$ , nothing requires  $\Box$  to have an S5 logic. By contrast, I will take metaphysical necessity  $\Box$  to be the strongest objective modality, and so an S5 logic is appropriate. Since the present aim is to develop a bimodal logic for tense and metaphysical modality, I will omit further consideration of the restricted modalities  $\Box$ ,  $\Box$ ,  $\Box$ , and  $\Box$ .

### 3.2 Bimodal Logic

Recall the propositional language  $\mathcal{L} = \langle \mathbb{L}, \perp, \rightarrow, \Box, \Box, \Box \rangle$  where  $\mathbb{L} := \{p_i \mid i \in \mathbb{N}\}$  is a set of *sentence letters* and the *well-formed sentences* of  $\mathcal{L}$  are defined as follows:

$$\varphi ::= p_i \mid \perp \mid \varphi \rightarrow \varphi \mid \Box\varphi \mid \Box\varphi \mid \Box\varphi.$$

Letting  $\mathcal{Q}\Gamma := \{\mathcal{Q}\gamma : \gamma \in \Gamma\}$  for any  $\mathcal{Q} \in \{\Box, \Box, \Box\}$  where  $\varphi_{\langle \Box, \Box \rangle}$  is the result of exchanging all occurrences of  $\Box$  and  $\Box$  in  $\varphi$ , the *Logic of Tense and Modality* **TM** is the smallest extension of the set of classical propositional tautologies **PL** to be closed under all instances of the following axiom and rule schemata:

|           |                                                                                                |           |                                                                                                       |
|-----------|------------------------------------------------------------------------------------------------|-----------|-------------------------------------------------------------------------------------------------------|
| <b>MP</b> | $\varphi, \varphi \rightarrow \psi \vdash \psi.$                                               | <b>TD</b> | <i>If <math>\vdash \varphi</math>, then <math>\vdash \varphi_{\langle \Box, \Box \rangle}</math>.</i> |
| <b>MK</b> | <i>If <math>\Gamma \vdash \varphi</math>, then <math>\Box\Gamma \vdash \Box\varphi</math>.</i> | <b>TK</b> | <i>If <math>\Gamma \vdash \varphi</math>, then <math>\Box\Gamma \vdash \Box\varphi</math>.</i>        |
| <b>MT</b> | $\Box\varphi \rightarrow \varphi.$                                                             | <b>TL</b> | $\Diamond\top \rightarrow (\Box\Box\varphi \rightarrow \Delta\varphi).$                               |
| <b>M4</b> | $\Box\varphi \rightarrow \Box\Box\varphi.$                                                     | <b>T4</b> | $\Box\varphi \rightarrow \Box\Box\varphi.$                                                            |
| <b>MB</b> | $\varphi \rightarrow \Box\Diamond\varphi.$                                                     | <b>TA</b> | $\varphi \rightarrow \Box\Diamond\varphi.$                                                            |
| <b>MF</b> | $\Box\varphi \rightarrow \Box\Box\varphi.$                                                     | <b>TF</b> | $\Box\varphi \rightarrow \Box\Box\varphi.$                                                            |

Whereas **MP**, **MT**, **M4**, and **MB** are familiar from propositional modal logic, **MK** combines the **K** axiom and the rule of necessitation **N** into a single metarule that distributes necessity over deduction. Similarly, **TK** distributes the future operator over deduction so that all deductions persist throughout the future. Since time is a total order, **TL** requires time to be left-linear where **TD** makes the logic symmetric with respect to the past and future at each time, thereby enforcing right-linearity as well. Additionally, **T4** requires the temporal ordering  $\leq$  to be transitive and **TA** asserts that the present is past to all future times. Were one to drop the temporal symmetry—

e.g., by assuming there is a first time but no last time— then **TD** must be given up where the appropriate duals of the axioms above must be added to the logic.

Whereas the axioms and rules discussed so far include either modal or temporal operators, **TF** and **MF** constrain the interaction between tense and modal operators. In particular, **TF** asserts that what is necessary is always going to be necessary and **MF** asserts that what is necessary is necessarily always going to be the case. Since  $\Box\Box\varphi \rightarrow \Box\varphi$  is an instance of **MT**, it follows from **MF** that  $\Box\varphi \rightarrow \Box\varphi$  by classical reasoning. Since  $\Box\varphi \rightarrow \Box\varphi$  follows by **TD**, we may take these results together with **MT** to conclude that  $\Box\varphi \rightarrow (\Box\varphi \wedge \varphi \wedge \Box\varphi)$  again by classical reasoning, and so:

$$\mathbf{P1} \quad \Box\varphi \rightarrow \Delta\varphi.$$

$$\mathbf{P2} \quad \nabla\varphi \rightarrow \Diamond\varphi.$$

Whereas **P1** follows from the definition of  $\Delta$ , **P2** is equivalent by classical logic. Thus the perpetuity principles follow from **MF** and **MT** by classical reasoning. In addition to being stated in primitive terms, **MF** and **MT** are easy to justify since whatever is necessary is always going to be necessary in addition to being actual.

Whereas **MF** plays a critical role in deriving the perpetuity principles, **TF** makes the tense and modal operators commute. Since  $\Box\varphi \rightarrow \Box\Box\varphi$  follows from **TF** by **TD**, we may derive  $\Box\varphi \rightarrow (\Box\Box\varphi \wedge \Box\varphi \wedge \Box\Box\varphi)$  where  $\Box\varphi \rightarrow \Box(\Box\varphi \wedge \varphi \wedge \Box\varphi)$  follows by modal reasoning. Thus  $\Box\varphi \rightarrow \Box\Delta\varphi$  follows from the definitions where  $\Delta$  is equivalent:

$$\mathbf{P3} \quad \Box\varphi \rightarrow \Box\Delta\varphi.$$

$$\mathbf{P4} \quad \Diamond\nabla\varphi \rightarrow \Diamond\varphi.$$

Since  $\Box\Diamond\varphi \rightarrow \Diamond\varphi$  is an instance of **MT**, it follows from **TK** and classical reasoning that  $\Box\Diamond\varphi \rightarrow \Box\Diamond\varphi$ . We may then derive  $\Diamond\varphi \rightarrow \Box\Diamond\varphi$  from **MB** and **M4** by standard modal reasoning where  $\Box\Diamond\varphi \rightarrow \Box\Diamond\varphi$  is an instance of **TF**. Thus  $\Diamond\varphi \rightarrow \Box\Diamond\varphi$  where  $\Diamond\varphi \rightarrow \Box\Diamond\varphi$  follows by **TD**, and so  $\Diamond\varphi \rightarrow (\Box\Diamond\varphi \wedge \Diamond\varphi \wedge \Box\Diamond\varphi)$  which is equivalent to  $\Diamond\varphi \rightarrow \Delta\Diamond\varphi$ . Given **P4**, we may draw as a conclusion where  $\Delta$  is equivalent:

$$\mathbf{P5} \quad \Diamond\nabla\varphi \rightarrow \Delta\Diamond\varphi.$$

$$\mathbf{P6} \quad \nabla\Box\varphi \rightarrow \Box\Delta\varphi.$$

Together with **P1** and **P2**, the theorems above begin to characterize the interactions between tense and modality in **TM**. A number of additional theorems will be derived in the §5. Although I will consider a weaker logic than **TM** in §4, the remainder of the present section will strengthen **TM** by imposing additional frame constraints.

### 3.3 Extensions

Since the logic for metaphysical modality already describes the strongest objective modality, it remains to strength the tense logic by further constraining the temporal order  $\mathcal{T}$ . In addition to taking  $\mathcal{T}$  to be a totally ordered abelian group, consider:

DISCRETE PAST: For any time  $x \in T$ , if there is an earlier time  $y < x$ , then there is a greatest earlier time  $y' < x$  where  $z \leq y'$  for all  $z < x$ .

DISCRETE FUTURE: For any time  $x \in T$ , if there is a later time  $y > x$ , then there is a least latter time  $y' > x$  where  $z \geq y'$  for all  $z > x$ .

DENSE: For any times  $x, y \in T$  where  $x < y$ , there is a time  $z \in T$  where  $x < z < y$ .

COMPLETE: Every set of times  $X \subseteq T$  bounded above has a least upper bound.

The frame constraints above impose restrictions on the temporal order  $\mathcal{T} = \langle T, +, \leq \rangle$  included in a frame. Corresponding to these alternatives are the following axioms:

$$\begin{array}{ll}
\mathbf{UP} & \Diamond \top. \\
\mathbf{BP} & \Box \perp \vee \Diamond \Box \perp. \\
\mathbf{DP} & (\Box \varphi \wedge \varphi \wedge \Diamond \top) \rightarrow \Diamond \Box \varphi. \\
\mathbf{DN} & \Box \Box \varphi \rightarrow \Box \varphi. \\
\mathbf{UF} & \Diamond \top. \\
\mathbf{BF} & \Box \perp \vee \Diamond \Box \perp. \\
\mathbf{DF} & (\Box \varphi \wedge \varphi \wedge \Diamond \top) \rightarrow \Diamond \Box \varphi. \\
\mathbf{CO} & \Delta(\Box \varphi \rightarrow \Diamond \Box \varphi) \rightarrow (\Box \varphi \rightarrow \Box \varphi).
\end{array}$$

Quantifying over just the *unbounded worlds*  $H_{\mathcal{F}}^u := \{\tau \in H_{\mathcal{F}} \mid \text{dom}(\tau) \text{ is unbounded}\}$  validates **UP** and **UF**. Alternatively, restricting quantification to the *bounded worlds*  $H_{\mathcal{F}}^b := \{\tau \in H_{\mathcal{F}} \mid \text{dom}(\tau) \text{ is bounded}\}$  validates **BP** and **BF**. Additionally, **DP** and **DF** correspond to the DISCRETE PAST and DISCRETE FUTURE constraints which require there to be an immediately preceding or succeeding time, respectively. In opposition to these discreteness axioms, **DN** and **CO** correspond to the DENSE and COMPLETE constraints on the temporal order. By assuming **TD** given previously, **UP**, **BP**, and **DP** are interderivable with **UF**, **BF** and **DF**, respectively. Although one could maintain certain axioms without their temporal duals by dropping **TD**, I will assume that time is symmetric throughout what follows for simplicity.

Since there is little sense in arguing for one axiom system over another independent of any particular application, I will focus on an extension **TM<sup>D</sup>** that includes **DN** and is valid over all models in which  $\mathcal{T}$  is a DENSE order. The formalization of **H1** considered in §2.1 is derivable from **DN** by **MK** where  $\Box$  is equivalent:

$$\begin{array}{ll}
\mathbf{P7} & \Box(\Box \Box \varphi \rightarrow \Box \varphi). \\
\mathbf{P8} & \Box(\Diamond \varphi \rightarrow \Diamond \Diamond \varphi).
\end{array}$$

I show in **T6** and **T7** that **DN** is invalid over all models of  $\mathcal{L}$  and valid over the models of  $\mathcal{L}$  with a DENSE temporal order. Restricting to the DENSE models of  $\mathcal{L}$ , it follows that both the density axiom **DN** and its dual are necessary. Given that there are models of  $\mathcal{L}$  in which **DN** and subsequently both **P7** and **P8** are invalid, one might deny that ‘ $\Box$ ’ expresses metaphysical modality on account of failing to include discrete worlds in which density fails to hold. The question is which class of models provides an adequate characterization of metaphysical modality as opposed to another more restricted modality. Controversies of this kind are familiar from modal metaphysics. For instance, whereas Salmon [27] insists that there are exceptions to **M4**, one might take Salmon to be describing a more restricted modality which, however familiar, is strictly weaker than metaphysical modality. Rather than attempting to settle such disputes here, what matters for present purposes is that disputes of this kind can be had with respect to **H1** that Dorr and Goodman claimed could not be captured without assuming a Montagovian semantics for the modal operator  $\Box$ .

Besides the various extensions of **TM** that one may consider, there are a number of further operators by which to extend the expressive power of  $\mathcal{L}$ . In particular, the

*next*  $\textcircled{F}$  and *previous*  $\textcircled{P}$  operators have natural applications given a restriction to the discrete frames over which **DF** and **DP** are valid. Consider the following semantics:

$$(\textcircled{F}) \quad \mathcal{M}, \tau, x \models \textcircled{F}\varphi \text{ iff } \mathcal{M}, \tau, y \models \varphi \text{ for some } y > x \text{ where } y \leq z \text{ for all times } z > x.$$

$$(\textcircled{P}) \quad \mathcal{M}, \tau, x \models \textcircled{P}\varphi \text{ iff } \mathcal{M}, \tau, y \models \varphi \text{ for some } y < x \text{ where } y \geq z \text{ for all times } z < x.$$

Citing Dana Scott, Prior [16, p. 66] introduces operators for *tomorrow* and *yesterday* which are represented here as  $\textcircled{F}$  and  $\textcircled{P}$  though does not provide a semantics. Instead, Prior focus on the *metric tense operators*  $Pn\alpha$  and  $Fn\alpha$  which indicate that  $\alpha$  occurs at a distance  $n$  from the time of evaluation in either the past or future. To incorporate metric tense operators,  $\mathcal{L}$  would need to include singular terms for durations which I will not explore here. By contrast, the following semantics for the *since*  $\triangleleft$  and *until*  $\triangleright$  operators introduced by Kamp [28] can be provided for a propositional language:

$$(\triangleleft) \quad \mathcal{M}, \tau, x \models \varphi \triangleleft \psi \text{ iff } \mathcal{M}, \tau, z \models \psi \text{ for some time } z < x \text{ where } \mathcal{M}, \tau, y \models \varphi \text{ for all intermediate times } y \in T \text{ where } z < y < x.$$

$$(\triangleright) \quad \mathcal{M}, \tau, x \models \varphi \triangleright \psi \text{ iff } \mathcal{M}, \tau, z \models \psi \text{ for some time } z > x \text{ where } \mathcal{M}, \tau, y \models \varphi \text{ for all intermediate times } y \in T \text{ where } x < y < z.$$

Whereas the operators  $\triangleleft$  and  $\triangleright$  have wide ranging applications that presume nothing of the structure of time,  $\textcircled{F}$  and  $\textcircled{P}$  are only worth including in a language used to study discrete systems. Nevertheless, there are many discrete systems for which  $\textcircled{F}$  and  $\textcircled{P}$  are both meaningful and natural to consider. For instance, the moves in a chess game make for a vivid example as do the clock cycles on a computer, sequences of program executions, or the successive actions of one or more agents in a system.

The power of the task semantics for  $\mathcal{L}$  lies in the combination of the primitive elements included in a frame in order to provide the construction of possible worlds. In particular, the task relation constrains the range of possible worlds by deciding which transitions between world states are possible over which durations. Given the temporal structure of each possible world, it is easy to shift the world state that a possible world occupies by shifting the time of evaluation without making times endogenous to the truth-conditions of the language. In addition to preserves traditional semantic clauses for the tense and modal operators, this approach strengthens the logic by validating the bimodal interaction principles **P1** – **P6**. The following section will close by reviewing the paradox of the open future and drawing out the connections between the present approach and non-deterministic dynamical systems theory.

## 4 Tense and Modality

The construction of possible worlds assumes that time is a total order. For any world  $\tau \in H_{\mathcal{F}}$  and time  $x \in T$  in a model  $\mathcal{M}$  of  $\mathcal{L}$ , there is a determinate past and future relative to  $x$  where every well-formed sentence of  $\mathcal{L}$  has a unique truth-value in  $\tau$  at  $x$ . However natural it may be to take the past to be determined up to any given moment, it is much more contentious to suppose that each time determines a unique future. Arguments as old as Aristotle have sought to lend credence to the idea that

the future differs from the past in remaining open to determination by admitting a range of incompatible future alternatives, none of which is singled out as the actual future. Thomason [17] brings this point out as follows:

[T]he basic issue here seems to be whether or not one is prepared to accept as meaningful the assertion that there is always, whether we know it or not, a single possible future which, from the perspective of a given time will be its actual future. (p. 270)

In considering future contingents such as ‘There will be a sea battle tomorrow’, many have sought to follow Aristotle in denying that such claims have a truth-value until the future comes about. Although there may be no fact of the matter whether there will be a sea battle tomorrow when it is considered today, we may nevertheless expect it to be settled tomorrow whether there ends up being a sea battle or not.

The construction of possible worlds takes each time in each possible world to have a complete past and future that is determined in every respect. However, evaluating a future contingent claim at a possible world and time does not carry any commitment that the world of evaluation is the actual world whose future is guaranteed to take place. By considering all worlds that intersect the world of evaluation at the time of evaluation, one may leave it open which of those worlds will play out. Just as one may consider the various chess games that may transpire from a given board state, it is also natural to consider the various routes between two points on a map, the different states a computer may transition through, or the different generation trajectories provided by an LLM. Despite making evaluations according to one possible world or another when considering the evolution of a system, doing so does not require any of the possible worlds to be the *actual* world that will take place. Even if none of the possible worlds are foretold to be actual, we can say in full confidence what each possible world includes. The following subsection will defend an approach of this kind by considering the shortcomings that face the alternatives. I will conclude in §4.2 by drawing connections to dynamical systems theory in contrast to the two-dimensional form which both Montague [1] and Kaplan’s [2] two-dimensional semantics maintain.

## 4.1 Open Future

One way to accommodate an open future at each time in every world is to weaken the tense logic **TM** by dropping **TL**.<sup>31</sup> Instead of requiring the times in each frame to be totally ordered as in §3, there is a tradition going back to Prior [16] and Kripke [20] that gives up totality by taking the times to be a connected left-linear partial order so that each time  $x \in T$  may admit multiple futures. Despite admitting a branching structure, we may nevertheless preserve the definition from §3 that a world history is any function  $\tau : X \rightarrow W$  where  $X \subseteq T$  is a nonempty convex set of times and  $\tau(x) \Rightarrow_y \tau(x+y)$  for all  $x, y \in T$  where  $x, x+y \in X$ . As before, I will continue to refer to world histories as possible worlds for familiarity and consistency.

This proposal faces the same problems brought out for the Peircean semantics considered above. For instance, consider a chess game  $\tau$  at a time  $x$  in which there is a future time  $y > x$  where the black king is in checkmate and an incomparable

<sup>31</sup>One might also give up **TD** in order to validate  $\Diamond\top \rightarrow (\Box\Box\varphi \rightarrow \Delta\varphi)$  which requires the past to be unique and then add the duals of the other axioms by swapping  $\Box$  and  $\Diamond$ .

future time  $z > x$  where the white king is in checkmate. By the semantics for the future tense operator, ‘White is going to win’ and ‘Black is going to win’ both come out true though this would seem impossible. Much more natural is to assert ‘White *could* win’ and ‘Black *could* win’ to indicate that there is some determinate future in which White wins and some determinate future in which Black wins. After all, these are the sorts of chess games that are still worth playing. As natural as this may seem, there is no easy way to achieve this if the modals quantify over possible worlds with open futures and the future tense operator quantifies over incomparable times that are later than the time of evaluation. Following the Ockhamist by introducing a further parameter at which to evaluate sentences that specifies the particular branch within a possible world returns the situation that we were in previously in which sentences were evaluated at parameters that suffice to determine a fixed past and future.

Consider a frame  $\mathcal{F} = \langle W, \mathcal{T}, \Rightarrow \rangle$  where  $\mathcal{T}$  is a totally ordered abelian group as before. The paradox of assuming the future to be determined for a given world-time pair turns on how the parameters at which the sentences of the language are understood. For instance, taking there to be a possible world  $@ \in H_{\mathcal{F}}$  that represent the *actual world* is committed to a determinate past and future for any given time in that world. We may indicate this commitment by including a semantic primitive for the actual world in the frame  $\mathcal{F}_{@} = \langle W, \mathcal{T}, \Rightarrow, @ \rangle$ . By contrast, the task semantics for  $\mathcal{L}$  does not posit an actual world  $@$  since it unnatural to do so without admitting that the future from any time in  $@$  has already been foretold.

Instead of positing an actual world  $@$ , a weaker commitment takes each frame  $\mathcal{F}_{\#} = \langle W, \mathcal{T}, \Rightarrow, \# \rangle$  to include a distinguished element  $\# \in W$  that represents the *present world state* that currently obtains.<sup>32</sup> Given  $\#$  and fixing a time  $n$ , we may consider all possible worlds that occupy this world state at  $n$  without maintaining the commitment that one of these worlds is already determined to be the actual future that transpires from  $\#$ . Rather, all of the possible worlds that occupy  $\#$  at  $n$  may be considered to be on a par where none may claim to represent the actual future any more than the others. Although far less committing than  $\mathcal{F}_{@}$ , the significance of a frame such as  $\mathcal{F}_{\#}$  is fleeting since what obtains now is only momentary. Instead of attempting to select different frames for different moments, no element of  $W$  has been designated as the present world state. As a result, the semantics for  $\mathcal{L}$  does not posit the structure required to say what is true *simpliciter*, but only what is true by the lights of one world and time or another where none are to be distinguished.

We may now return to the chess game from before in which both Black and White have a chance at winning. Given the state of the board after 14 moves, there simply is no actual game with a determinate future that proceeds from the present board state. Rather, we may consider various chess games that proceed from the present board state at move 14. By taking turns choosing which tasks to enact, the players compete in their abilities to determine which game of chess becomes actual. Rather than building indeterminacy into the temporal structure itself by taking time to be only partially ordered, the task semantics for  $\mathcal{L}$  locates indeterminacy in the space of possible state transitions between world states. Letting time be a total order, each

---

<sup>32</sup>Alternatively, one might designate a *past* that includes the world states that have obtained up to and including the world state that currently obtains, echoing growing block theories of time.

world history  $\tau \in H_{\mathcal{F}}$  represents one possible path through the space of world states, constrained by which transitions the task relation permits. Different world histories from the same world state represent genuine alternatives about how the system might evolve without requiring multiple incomparable future times. So long as none of these possible worlds are assumed to be actual, no harm comes in allowing each possible world to specify a determinate past and future from any given time.

Something similar may be said for Aristotle's sea battle. Since some of the possible worlds intersecting the present world state include a sea battle on the following day and other intersecting possible worlds do not, it is contingent whether there is going to be a sea battle. In symbols, we may say  $\Diamond p_s \wedge \Diamond \neg p_s$  where  $p_s$  expresses that a sea battle is taking place. So long as no possible world is designated as the actual world, we may take it to be open which of these worlds will become actual. Of course, by the time tomorrow comes we will know whether there has been a sea battle or not, and so nothing regarding the sea battle will remain left to determine. Although the present is all that obtains, one might also claim that there is a determinate past consisting of those states that have obtained, or at least so it may seem. However tempting, the semantics does not posit an actual past any more than it posits an actual future.

## 4.2 Dynamical Systems

Dynamical systems theory provides a general mathematical framework for modeling the possible evolutions of a system, suggesting a natural connection with tense and modal reasoning. Whereas the frames defined above take time to have both group and order structure, non-deterministic dynamical systems typically consist of a set of states  $W$ , a monoid  $\langle T, + \rangle$  for positive durations, and a set of relations  $\{R_x\}_{x \in T}$  indexed by  $T$  that satisfy the following conditions for all  $w, u, v \in W$  and  $x, y \in T$ :<sup>33</sup>

*Nullity:*  $R_0(w, w)$ .

*Compositionality:* If  $R_x(w, u)$  and  $R_y(u, v)$ , then  $R_{x+y}(w, v)$ .

Amounting to no more than a change of notation,  $R_x(w, u)$  expresses with an indexed family of two-place relations what  $w \Rightarrow_x u$  expresses with a three-place relation. However, in order to provide semantic clauses for tense operators which quantify over all times that are *earlier than* or *later than* a given time of evaluation, an order must also be provided. Since it is just as natural to subtract times as it is to add them, I will take every time  $x$  to have an inverse  $-x$ , writing  $x - y$  in place of  $x + (-y)$  as usual. Moreover, the order of addition does not make a difference, motivating commutativity  $x + y = y + x$  for all times. Accordingly, I will restrict attention to dynamical systems in which the times  $\mathcal{T} = \langle T, +, \leq \rangle$  form a totally ordered abelian group as before.

A dynamical system is *deterministic* if  $R_x$  is functional for all  $x \in T$ , justifying the notation for the *world functions*  $\{f_x\}_{x \in \mathcal{T}_M}$  where  $f_x(w) = u$  just in case  $R_x(w, u)$ . A *deterministic model* is any  $\mathcal{D} = \langle W, \mathcal{T}_M, \{f_x\}_{x \in T}, |\cdot| \rangle$  where  $|p_i| \subseteq W$  for all  $p_i \in \mathbb{L}$  and  $\langle W, \mathcal{T}_M, \{f_x\}_{x \in T} \rangle$  is a deterministic dynamical system. Instead of constructing possible worlds from world states, tasks, and times, Williamson [29] provides a semantics for

<sup>33</sup>A *monoid* is any  $\langle M, \cdot \rangle$  where: (1)  $a \cdot b \in M$  whenever  $a, b \in M$ ; (2)  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  for all  $a, b, c \in M$ ; and (3)  $1_M \in M$  where  $a \cdot 1_M = 1_M \cdot a = a$  for all  $a \in M$ .

a higher-order bimodal language in terms of the primitive world functions included in a deterministic model. I will present a propositional fragment as follows:

- $(p_i) \mathcal{D}, w \models p_i \text{ iff } w \in |p_i|.$
- $(\perp) \mathcal{D}, w \not\models \perp.$
- $(\rightarrow) \mathcal{D}, w \models \varphi \rightarrow \psi \text{ iff } \mathcal{D}, w \not\models \varphi \text{ or } \mathcal{D}, w \models \psi.$
- $(\Box) \mathcal{D}, w \models \Box\varphi \text{ iff } \mathcal{D}, u \models \varphi \text{ for all } u \in W.$
- $(\Box) \mathcal{D}, w \models \Box\varphi \text{ iff } \mathcal{D}, f_x(w) \models \varphi \text{ for all } x \in T \text{ where } x < 0.$
- $(\Box) \mathcal{D}, w \models \Box\varphi \text{ iff } \mathcal{D}, f_x(w) \models \varphi \text{ for all } x \in T \text{ where } x > 0.$

Instead of interpreting sentences at a model, possible world, and time, the semantic clauses above evaluate sentences directly at a model and world state. This approach may claim the advantage of validating the perpetuity principles **P1** – **P6** from before without undermining the intelligibility of the truth-conditions for the sentences of the language. Nevertheless, the semantics above is restricted to deterministic systems, and so unable to model systems in which states fail to determine a unique past and future. For instance, the board states in a game of chess do not typically mandate any particular continuation, leaving open many different incompatible futures.

Since there is no latitude in the world functions for any contingency among possible futures, it remains to weaken the temporal order in attempt to encode an open future. As already observed, taking  $\mathcal{T}$  to be a partial order returns the paradox facing the Peircean semantics which quantifies over all or some incomparable future times. For instance, it is absurd to admit that both Black and White are going to win, or to claim that White will win and lose whenever there is future contingency. Nevertheless, for deterministic systems, a Peircean semantics is perfectly appropriate. For instance, the motions of the planets are well modeled by a deterministic system constrained under Newton's laws. Given the total state of the solar system including the position and momentum of each planet, all future states of that system are fully determined. The construction of possible worlds is not required to model deterministic systems since if world states determine their past and future, there is no need to fix a temporal parameter. Even so, these are small gains in compensation for the ability to represent non-deterministic systems, especially when metaphysical modality is under discussion. If there are systems whose world states have incompatible futures, the semantics for the strongest modality must take into consideration all possible futures.

More fundamental than parameterization is the cost of the Ockhamist strategy to add an evaluation parameter by which to distinguish the different futures at any given world state. Although one could include a family of evaluation functions in the point of evaluation, doing so multiplies the range of primitive functions without putting them in any meaningful correspondence. Rather than evaluating sentences at a world state and a family of world functions for the modals to quantify over, possible worlds are parameterized by times in order to refer to counterpart times across possible worlds. For instance, having blundered the dark square bishop, Black may lament the missed opportunity to have won much sooner. By taking possible worlds to be functions from

times to world states and evaluating sentences at a possible world and time, the task semantics takes times to provide a reference that can be used to identify world states in different possible worlds. As brought out above, temporal parameters may fall outside the domain of a possible world for the sake of comparison to other intersecting and non-intersecting possible worlds. Given the closure of the group structure for times, possible worlds come in abundance so that every possible world time-shifted into the past or future exists. Thus by quantifying over all possible worlds, the modal operators reach every world state however disconnected. By contrast, the semantics above takes the modals to quantify directly over world states, achieving generality much more directly but without leaving room for contingency in the past or future.

Drawing the connections between the present semantic framework and dynamical systems theory demonstrates that these ideas are not new. Indeed, it has been standard to represent the evolution of a system with functions since the time of Galileo, Newton, and Leibniz in the seventeenth century. It is unfathomable not just in physics but throughout the sciences to represent the various evolutions of a system by primitive points. The semantics for bimodal languages with tense and modal operators is no different. Thomason [30] brings out this point as follows:

Physics should have helped us to realize that a temporal theory of a phenomenon  $X$  is, in general, more than a simple combination of two components: the statics of  $X$  and the ordered set of temporal instants. The case in which all functions from times to world-states are allowed is uninteresting; there are too many such functions, and the theory has not begun until we have begun to restrict them. [...] The general moral, then, is that we shouldn't expect the theory of time +  $X$  to be obtained by mechanically combining the theory of time and the theory of  $X$ .  
(p. 135)

Montague [1] and Kaplan's [2] two-dimensional semantics assumes a dynamics that predates the development of functional dynamics during the Scientific Revolution. Instead of attempting to salvage these theories while evaluating sentences at primitive worlds and times, the construction of possible worlds preserves the spirit of what these theories while strengthening the logic. Although dynamical systems are nothing new, what has not been adequately explored is the connection between functional dynamics and bimodal reasoning. This paper provides a step in that direction.

## 5 Appendix

### 5.1 Model Theory

This section concerns the two-dimensional models introduced in §2.1. Definitions will be restated for convenience throughout.

**D1** A *two-dimensional model* is a structure  $\mathcal{M} = \langle W, T, \leq, |\cdot| \rangle$  where:

*Worlds:* A nonempty set of *worlds*  $W$ .

*Times:* A nonempty set of *times*  $T$ .

*Order:* A weak total order  $\leq$  on  $T$ .

*Interpretation:* A function  $|p_i| \subseteq W \times T$  is a set of world-time pairs for each  $p_i \in \mathbb{L}$ .

**D2** The language  $\mathcal{L}^M := \langle \mathbb{L}, \perp, \rightarrow, \boxtimes, \boxplus, \boxminus \rangle$  where  $\mathbb{L} := \{p_i : i \in \mathbb{N}\}$  is a countable set of sentence letters where the remaining symbols denote falsity, material implication, the disputed modal operator that Thomason reads ‘necessarily always’, the universal past tense operator, and the universal future tense operator, respectively.

**D3** Truth in a two-dimensional model at a world and time is defined recursively:

- $(p_i)$   $\mathcal{M}, w, x \models p_i$  iff  $\langle w, x \rangle \in |p_i|$ .
- $(\perp)$   $\mathcal{M}, w, x \not\models \perp$ .
- $(\rightarrow)$   $\mathcal{M}, w, x \models \varphi \rightarrow \psi$  iff  $\mathcal{M}, w, x \not\models \varphi$  or  $\mathcal{M}, w, x \models \psi$ .
- $(\boxtimes)$   $\mathcal{M}, w, x \models \boxtimes \varphi$  iff  $\mathcal{M}, u, y \models \varphi$  for all  $u \in W$  and  $y \in T$ .
- $(\boxplus)$   $\mathcal{M}, w, x \models \boxplus \varphi$  iff  $\mathcal{M}, w, y \models \varphi$  for all  $y \in T$  where  $y \leq x$ .
- $(\boxminus)$   $\mathcal{M}, w, x \models \boxminus \varphi$  iff  $\mathcal{M}, w, y \models \varphi$  for all  $y \in T$  where  $x \leq y$ .

**D4** A nonempty relation  $Z \subseteq (W_1 \times T_1) \times (W_2 \times T_2)$ — writing  $(w, x) \rightarrow (v, y)$  to represent pairs in  $Z$ — is a  $\mathcal{L}^M$ -bisimulation between  $\mathcal{M}_1$  and  $\mathcal{M}_2$  just in case whenever  $(w, x) \rightarrow (w', x')$ , all of the following conditions hold:

- Atomic harmony:* For all  $p_i \in \mathbb{L}$ ,  $\mathcal{M}_1, w, x \models p_i$  just in case  $\mathcal{M}_2, w', x' \models p_i$ .
- Past Forth:* For every  $y$  with  $y \leq x$  there is  $y'$  with  $y' \leq x'$  and  $(w, y) \rightarrow (w', y')$ .
- Past Back:* For every  $y'$  with  $y' \leq x'$  there is  $y$  with  $y \leq x$  and  $(w, y) \rightarrow (w', y')$ .
- Future Forth:* For every  $y$  with  $x \leq y$  there is  $y'$  with  $x' \leq y'$  and  $(w, y) \rightarrow (w', y')$ .
- Future Back:* For every  $y'$  with  $x' \leq y'$  there is  $y$  with  $x \leq y$  and  $(w, y) \rightarrow (w', y')$ .
- Global Forth:* For every  $(u, z) \in W_1 \times T_1$  there is  $(u', z') \in W_2 \times T_2$  with  $(u, z) \rightarrow (u', z')$ .
- Global Back:* For every  $(u', z') \in W_2 \times T_2$  there is  $(u, z) \in W_1 \times T_1$  with  $(u, z) \rightarrow (u', z')$ .

**L1** If  $Z$  is a  $\mathcal{L}^M$ -bisimulation between  $\mathcal{M}_1$  and  $\mathcal{M}_2$  and  $(w, x) \rightarrow (w', x')$ , then for every  $\mathcal{L}^M$ -formula  $\varphi$ ,  $\mathcal{M}_1, w, x \models \varphi$  just in case  $\mathcal{M}_2, w', x' \models \varphi$ .

*Proof.* The proof proceeds by induction on the complexity of  $\varphi$  where the case for the sentence letters and extensional operators are routine.

*Case  $\boxminus$ :* Assume  $\varphi = \boxminus \psi$ . Supposing for contraposition that  $\mathcal{M}_2, w', x' \not\models \boxminus \psi$ , it follows that  $\mathcal{M}_2, w', y' \not\models \psi$  for some  $x' < y'$ . By *Future Back*, there is a  $y$  with  $x \leq y$  where  $(w, y) \rightarrow (w', y')$ , and so  $\mathcal{M}_1, w, y \not\models \psi$  by hypothesis. Thus  $\mathcal{M}_1, w, x \not\models \boxminus \psi$ . Contraposition and parity of reasoning establish that  $\mathcal{M}_1, w, x \models \boxminus \psi$  just in case  $\mathcal{M}_2, w', x' \models \boxminus \psi$ . The other tense operators are similar.

*Case  $\boxtimes$ :* Assume  $\varphi = \boxtimes\psi$ . Supposing for contraposition that  $\mathcal{M}_2, w', x' \not\models \boxtimes\psi$ , it follows that  $\mathcal{M}_2, u', z' \not\models \psi$  for some  $u' \in W_2$  and  $z' \in T_2$ . By *Global Back*, there is  $(u, z) \in W_1 \times T_1$  with  $(u, z) \rightarrow (u', z')$ , and so  $\mathcal{M}_1, u, z \not\models \psi$  by hypothesis. Thus  $\mathcal{M}_1, w, x \not\models \boxtimes\psi$ , where contraposition and parity of reasoning complete the case.

By induction on complexity, we may conclude that  $\mathcal{M}_1, w, x \models \varphi$  just in case  $\mathcal{M}_2, w', x' \models \varphi$  for all well-formed sentences  $\varphi$  in  $\mathcal{L}^M$ .  $\square$

**T1**  $\square$  is not definable in  $\mathcal{L}^M$ .

*Proof.* Assume for contradiction that  $\square$  is definable in  $\mathcal{L}^M$  so that  $\square\varphi$  abbreviates a well-formed sentence of  $\mathcal{L}^M$  with the following derived semantic clause:

$(\square) \mathcal{M}, w, x \models \square\varphi$  iff  $\mathcal{M}, u, x \models \varphi$  for all  $u \in W$ .

We define the models  $\mathcal{M}_1 = \langle W_1, T_1, \leq, |\cdot|_1 \rangle$  and  $\mathcal{M}_2 = \langle W_2, T_2, \leq, |\cdot|_2 \rangle$  where:

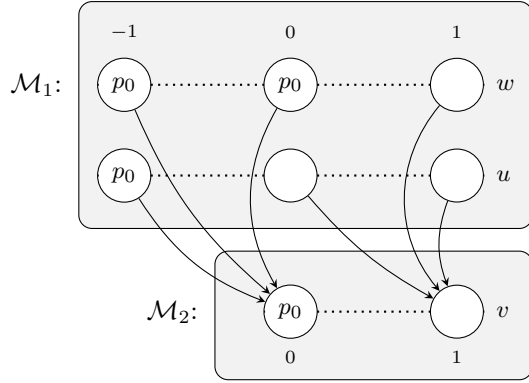
$\mathcal{M}_1$ :  $W_1 = \{w, u\}$ ,  $T_1 = \{-1, 0, 1\}$ , and  $|p_0|_1 = \{(w, -1), (w, 0), (u, -1)\}$ , so that  $p_0$  is true at  $(w, -1)$ ,  $(w, 0)$ , and  $(u, -1)$  in  $\mathcal{M}_1$ .

$\mathcal{M}_2$ :  $W_2 = \{v\}$ ,  $T_2 = \{0, 1\}$ , and  $|p_0|_2 = \{(v, 0)\}$ , so that  $p_0$  is true at  $(v, 0)$  in  $\mathcal{M}_2$ .

We may then define the relation  $Z \subseteq (W_1 \times T_1) \times (W_2 \times T_2)$  as depicted in the following diagram, writing  $(w, x) \rightarrow (v, y)$  to represent pairs in  $Z$  as before:

The relation  $Z$  consists of the following six pairs:

- $(w, -1) \rightarrow (v, 0),$
- $(w, 0) \rightarrow (v, 0),$
- $(w, 1) \rightarrow (v, 1),$
- $(u, -1) \rightarrow (v, 0),$
- $(u, 0) \rightarrow (v, 1),$
- $(u, 1) \rightarrow (v, 1).$



We verify that  $Z$  is a  $\mathcal{L}^M$ -bisimulation by checking the clauses of D4:

*Atomic harmony:* The pairs  $(w, -1) \rightarrow (v, 0)$ ,  $(w, 0) \rightarrow (v, 0)$ , and  $(u, -1) \rightarrow (v, 0)$  relate points that both satisfy  $p_0$ , while the pairs  $(w, 1) \rightarrow (v, 1)$ ,  $(u, 0) \rightarrow (v, 1)$ , and  $(u, 1) \rightarrow (v, 1)$  relate points that both fail to satisfy  $p_0$ . Thus for every related pair  $(w', x) \rightarrow (u', y)$  we have  $\mathcal{M}_1, w', x \models p_0$  just in case  $\mathcal{M}_2, u', y \models p_0$ .

*Past Forth:* For each pair  $(w', x) \rightarrow (u', y)$  in  $Z$ , every time  $z \leq x$  in  $T_1$  has some corresponding time  $z' \leq y$  in  $T_2$  with  $(w', z) \rightarrow (u', z')$ . For instance, we may observe that since  $(u, 0) \rightarrow (v, 1)$  and  $-1 \leq 0$ , we have  $(u, -1) \rightarrow (v, 0)$  where  $0 \leq 1$ .

*Past Back:* For each pair  $(w', x) \rightarrow (u', y)$  in  $Z$ , every time  $z \leq y$  in  $T_2$  has some corresponding time  $z' \leq x$  in  $T_1$  with  $(w', z') \rightarrow (u', z)$ . For instance, we may observe that since  $(u, 0) \rightarrow (v, 1)$  and  $0 \leq 1$ , we have  $(u, -1) \rightarrow (v, 0)$  where  $-1 \leq 0$ .

*Future Forth and Future Back:* Similar reasoning applies.

*Global Forth:* Every point of  $\mathcal{M}_1$  appears as the left-projection of some pair in  $Z$ .

*Global Back:* Every point of  $\mathcal{M}_2$  appears as the right-projection of some pair in  $Z$ .

Thus  $Z$  is a  $\mathcal{L}^M$ -bisimulation. Given that  $(w, 0) \rightarrow (v, 0)$ , it follows by [L1](#) that for all well-formed  $\varphi$  of  $\mathcal{L}^M$ , the following biconditional holds:

$$\mathcal{M}_1, w, 0 \models \varphi \text{ just in case } \mathcal{M}_2, v, 0 \models \varphi. \quad (*)$$

On the assumption that  $\Box$  is definable in  $\mathcal{L}^M$ , we may conclude that  $\mathcal{M}_1, w, 0 \models \Box p_0$  just in case  $\mathcal{M}_2, v, 0 \models \Box p_0$ . By the claimed semantic clause for  $\Box$  we have:

$$\begin{aligned} \mathcal{M}_1, w, 0 \models \Box p_0 &\Leftrightarrow \mathcal{M}_1, w, 0 \models p_0 \text{ and } \mathcal{M}_1, u, 0 \models p_0 \\ \mathcal{M}_2, v, 0 \models \Box p_0 &\Leftrightarrow \mathcal{M}_2, v, 0 \models p_0 \end{aligned}$$

Since  $\mathcal{M}_1, u, 0 \not\models p_0$ , it follows that  $\mathcal{M}_1, w, 0 \not\models \Box p_0$ . Given that  $v$  is the only world in  $W_2$  and  $\mathcal{M}_2, v, 0 \models p_0$ , it follows that  $\mathcal{M}_2, v, 0 \models \Box p_0$ . Hence  $\mathcal{M}_1, w, 0 \not\models \Box p_0$  while  $\mathcal{M}_2, v, 0 \models \Box p_0$ , contradicting  $(*)$ . Thus we may conclude by *reductio* that the operator  $\Box$  with the derived semantic clause given above is not definable in  $\mathcal{L}^M$ .  $\square$

**D5** Dorr and Goodman [\[15\]](#) compensate for the limited expressive power of  $\mathcal{L}^M$  by adding a countable set of time variables  $\mathcal{V} := \{t_i : i \in \mathbb{N}\}$  which may be bound by first-order quantifiers, thereby obtaining the language  $\mathcal{L}^F$ . An *assignment* is a function  $g : \mathcal{V} \rightarrow T$  from time variables in  $\mathcal{V}$  to times in  $T$  which is used to extend the semantics. Truth in a two-dimensional model at a world and time relative to an assignment extends the semantics for  $\mathcal{L}^M$  to include:

- $(\exists t) \mathcal{M}, w, x, g \models \exists t \varphi$  iff  $\mathcal{M}, w, x, g' \models \varphi$  for some  $g'$  differing from  $g$  at most in  $t$ .
- $(\text{Present}) \mathcal{M}, w, x, g \models \text{Present}(t)$  iff  $g(t) = x$ .

**L2** In  $\mathcal{L}^F$ ,  $\Box$  may be defined by  $\Box \varphi := \exists t [\text{Present}(t) \wedge \Box(\text{Present}(t) \rightarrow \varphi)]$ .

*Proof.* To establish that  $\Box$  is definable, we show:

$$\begin{aligned} \mathcal{M}, w, x, g \models \exists t [\text{Present}(t) \wedge \Box(\text{Present}(t) \rightarrow \varphi)] \\ \Leftrightarrow \text{there is } g' \text{ with } g'(t) = x \text{ and } \mathcal{M}, u, y, g' \models \text{Present}(t) \rightarrow \varphi \text{ for all } u \in W, y \in T \\ \Leftrightarrow \text{there is } g' \text{ with } g'(t) = x \text{ and } \mathcal{M}, u, x, g' \models \varphi \text{ for all } u \in W \\ \Leftrightarrow \mathcal{M}, u, x, g \models \varphi \text{ for all } u \in W \end{aligned}$$

The first equivalence follows from the semantics for  $\exists$ ,  $\wedge$ , and  $\boxtimes$ . For the second equivalence, since  $g'(x) = x$ , we have  $\mathcal{M}, u, y, g' \models \text{Present}(t) \rightarrow \varphi$  just in case either  $y \neq x$  or  $\mathcal{M}, u, y, g' \models \varphi$  with  $y = x$ . It follows that  $\mathcal{M}, u, y, g' \models \text{Present}(t) \rightarrow \varphi$  for all  $u \in W$  and  $y \in T$  just in case  $\mathcal{M}, u, x, g' \models \varphi$  for all  $u \in W$ . The final equivalence holds because  $t$  does not occur free in  $\varphi$ , so  $g$  and  $g'$  agree on the truth of  $\varphi$ , and so we may choose any  $g'$ -variant of  $g$  with  $g'(t) = x$  to witness the existential.  $\square$

**T2** Given the two-dimensional semantics for  $\mathcal{L}^F$ , both **P1** and **P2** are invalid.

*Proof.* Let  $\mathcal{M} = \langle W, T, \leq, |\cdot| \rangle$  be a two-dimensional model with worlds  $W = \{w, u\}$ , times  $T = \{0, 1\}$  where  $\leq$  is the usual order on  $\{0, 1\}$ , and  $|p_0| = \{(w, 0), (u, 0)\}$ . Since  $(w, 0) \in |p_0|$  and  $(u, 0) \in |p_0|$ , we have  $\mathcal{M}, w, 0, g \models p_0$  and  $\mathcal{M}, u, 0, g \models p_0$  where  $g$  is any assignment. It follows by **L2** that  $\mathcal{M}, w, 0, g \models \Box p_0$ .

However,  $\mathcal{M}, w, 0, g \models \Delta p_0$  just in case  $\mathcal{M}, w, y, g \models p_0$  for all  $y \in T$  with  $0 \leq y$ . Since  $0 \leq 1$  and  $(w, 1) \notin |p_0|$ , we have  $\mathcal{M}, w, 1, g \not\models p_0$ . Therefore  $\mathcal{M}, w, 0, g \not\models \Delta p_0$ . It follows that  $\mathcal{M}, w, 0, g \not\models \Box p_0 \rightarrow \Delta p_0$ , and so **P1** is invalid. Since **P2** is equivalent to **P1**, it follows that **P2** is also invalid.  $\square$

**D6** An order automorphism on the structure  $\langle T, \leq \rangle$  is a bijective function  $\bar{a} : T \rightarrow T$  such that for all  $x, y \in T$ , we have  $x \leq y$  just in case  $\bar{a}(x) \leq \bar{a}(y)$ .

**D7** Given a two-dimensional model  $\mathcal{M} = \langle W, T, \leq, |\cdot| \rangle$ , the worlds  $w, w' \in W$  are *time-shifted from  $y$  to  $x$* —written  $w \approx_y^x w'$ —if and only if there exists an order automorphism  $\bar{a} : T \rightarrow T$  where  $y = \bar{a}(x)$  and for all sentence letters  $p_i \in \mathbb{L}$  and times  $z \in T$ , we have  $\langle w, \bar{a}(z) \rangle \in |p_i|$  just in case  $\langle w', z \rangle \in |p_i|$ .

**L3** If  $w \approx_{\bar{a}(z)}^z u$  for some  $z \in T$ , then  $w \approx_{\bar{a}(z')}^{z'} u$  for all  $z' \in T$ .

*Proof.* Suppose  $w \approx_{\bar{a}(z)}^z u$  for some  $z \in T$ . By **D7**, there exists an order automorphism  $\bar{a} : T \rightarrow T$  where for all sentence letters  $p_i \in \mathbb{L}$  and all times  $x \in T$ :

$$\langle w, \bar{a}(x) \rangle \in |p_i| \text{ just in case } \langle u, x \rangle \in |p_i|. \quad (*)$$

Let  $z' \in T$  be arbitrary. Since  $\bar{a}(z') = \bar{a}(z')$  holds trivially,  $w \approx_{\bar{a}(z')}^{z'} u$  follows from  $(*)$ , where generalizing on  $z' \in T$  completes the proof.  $\square$

**D8** A two-dimensional model  $\mathcal{M} = \langle W, T, \leq, |\cdot| \rangle$  of  $\mathcal{L}$  is *abundant* just in case for every  $w \in W$  and times  $x, y \in T$ , there is some world  $w' \in W$  where  $w \approx_y^x w'$ .

**L4**  $\mathcal{M}_2, w, y \models \varphi$  just in case  $\mathcal{M}_2, u, x \models \varphi$  for any well-formed sentence  $\varphi$  of  $\mathcal{L}$  and abundant two-dimensional model  $\mathcal{M}_2 = \langle W, T, \leq, |\cdot| \rangle$  where  $w \approx_y^x u$ .

*Proof.* Assume  $w \approx_y^x u$  in  $\mathcal{M}_2 = \langle W, T, \leq, |\cdot| \rangle$ . The proof proceeds by induction on the complexity of  $\varphi$ . By **D7**, there exists an order automorphism  $\bar{a} : T \rightarrow T$  where

$y = \bar{a}(x)$  and for all sentence letters  $p_i \in \mathbb{L}$  and times  $z \in T$ :

$$\langle w, \bar{a}(z) \rangle \in |p_i| \Leftrightarrow \langle u, z \rangle \in |p_i|. \quad (\dagger)$$

*Base Case* ( $\varphi = p_i$ ): By  $(\dagger)$  with  $z = x$ , it follows that  $\langle w, \bar{a}(x) \rangle \in |p_i|$  just in case  $\langle u, x \rangle \in |p_i|$ . Since  $y = \bar{a}(x)$ , this gives  $\langle w, y \rangle \in |p_i|$  just in case  $\langle u, x \rangle \in |p_i|$ . By the semantic clause for sentence letters,  $\mathcal{M}_2, w, y \models p_i$  just in case  $\mathcal{M}_2, u, x \models p_i$ .

*Base Case* ( $\varphi = \perp$ ): By the semantic clause for  $\perp$ , we have  $\mathcal{M}_2, w, y \not\models \perp$  and  $\mathcal{M}_2, u, x \not\models \perp$ . Therefore the biconditional holds trivially.

*Inductive Case* ( $\varphi = \psi \rightarrow \chi$ ):

$$\begin{aligned} \mathcal{M}_2, w, y \models \psi \rightarrow \chi &\Leftrightarrow \mathcal{M}_2, w, y \not\models \psi \text{ or } \mathcal{M}_2, w, y \models \chi \\ &\Leftrightarrow \mathcal{M}_2, u, x \not\models \psi \text{ or } \mathcal{M}_2, u, x \models \chi \\ &\Leftrightarrow \mathcal{M}_2, u, x \models \psi \rightarrow \chi \end{aligned}$$

The induction hypothesis justifies the second equivalence when applied to  $\psi$  and  $\chi$ .

*Inductive Case* ( $\varphi = \Box\psi$ ):

$$\begin{aligned} \mathcal{M}_2, w, y \not\models \Box\psi &\Leftrightarrow \mathcal{M}_2, v, y \not\models \psi \text{ for some } v \in W \\ &\Leftrightarrow \mathcal{M}_2, v', x \not\models \psi \text{ for some } v \in W \\ &\Leftrightarrow \mathcal{M}_2, u, x \not\models \Box\psi \end{aligned}$$

Since  $\mathcal{M}_2$  is abundant where  $v \in W$  and  $x, y \in T$ , there is some  $v' \in W$  where  $v \approx_y^x v'$ . It follows by the induction hypothesis that  $\mathcal{M}_2, v', x \not\models \psi$ , where parity of reasoning establishes the converse. This justifies the second biconditional.

*Inductive Case* ( $\varphi = \Box\psi$ ):

$$\begin{aligned} \mathcal{M}_2, w, y \not\models \Box\psi &\Leftrightarrow \mathcal{M}_2, w, z \not\models \psi \text{ for some } z < y \\ &\Leftrightarrow \mathcal{M}_2, u, z' \not\models \psi \text{ for some } z' < x \\ &\Leftrightarrow \mathcal{M}_2, u, x \not\models \Box\psi \end{aligned}$$

For the forward direction of the second biconditional, assume that  $\mathcal{M}_2, w, z \not\models \psi$  for some  $z < y$ . Since  $\bar{a}$  is an order automorphism,  $\bar{a}^{-1}(z) < \bar{a}^{-1}(y)$  where  $\bar{a}^{-1}(y) = x$ . Letting  $z' = \bar{a}^{-1}(z)$ , we have  $z' < x$ . Since  $w \approx_y^x u$  where  $y = \bar{a}(x)$  and  $z = \bar{a}(z')$ , it follows by **L3** that  $w \approx_z^{z'} u$ . Applying the induction hypothesis to  $\psi$ , we have  $\mathcal{M}_2, w, z \models \psi$  just in case  $\mathcal{M}_2, u, z' \models \psi$ , and so  $\mathcal{M}_2, u, z' \not\models \psi$  for some  $z' < x$ .

For the backward direction of the second biconditional, assume  $\mathcal{M}_2, u, z' \not\models \psi$  for some  $z' < x$ . Since  $\bar{a} : T \rightarrow T$  is an order automorphism, we have  $\bar{a}(z') < \bar{a}(x)$ . Letting  $z = \bar{a}(z')$ , we have  $z < y$  given that  $y = \bar{a}(x)$ . Since  $w \approx_y^x u$ , we have  $w \approx_z^{z'} u$  by **L3**, and so  $\mathcal{M}_2, w, z \models \psi$  just in case  $\mathcal{M}_2, u, z' \models \psi$  by the induction hypothesis. Thus we may conclude that  $\mathcal{M}_2, w, z \not\models \psi$  for some  $z < y$ , completing the case.

*Inductive Case* ( $\varphi = \Box\psi$ ): The proof is analogous to the case for  $\Box\psi$ , using  $x \leq z$  and  $y \leq z'$  in place of  $z \leq x$  and  $z' \leq y$ .  $\square$

**T3** Both **P1** and **P2** are valid over the abundant two-dimensional models of  $\mathcal{L}$ .

*Proof.* Let  $\mathcal{M} = \langle W, T, \leq, |\cdot| \rangle$  be an abundant two-dimensional model of  $\mathcal{L}$ . Assume for *reductio* that  $\mathcal{M}, w, x \not\models \Box\varphi \rightarrow \Delta\varphi$  for some  $w \in W$ ,  $x \in T$ , and well-formed sentence  $\varphi$  of  $\mathcal{L}$ . Thus  $\mathcal{M}, w, x \models \Box\varphi$  and  $\mathcal{M}, w, x \not\models \Delta\varphi$ .

By the semantic clause for  $\Box$ , we have  $\mathcal{M}, u, x \models \varphi$  for all  $u \in W$ . In particular,  $\mathcal{M}, w, x \models \varphi$ . Since  $\Delta\varphi := \Box\varphi \wedge \varphi \wedge \Box\varphi$  and  $\mathcal{M}, w, x \models \varphi$ , it follows that either  $\mathcal{M}, w, x \not\models \Box\varphi$  or  $\mathcal{M}, w, x \not\models \Box\varphi$ .

*Case 1:* Assume  $\mathcal{M}, w, x \not\models \Box\varphi$ . By the semantic clause for  $\Box$ , there exists  $y \in T$  with  $y < x$  such that  $\mathcal{M}, w, y \not\models \varphi$ . Since  $\mathcal{M}$  is abundant, there exists  $w' \in W$  such that  $w \approx_y^x w'$ . By **L4**,  $\mathcal{M}, w, y \models \varphi$  just in case  $\mathcal{M}, w', x \models \varphi$ . Since  $\mathcal{M}, w, y \not\models \varphi$ , we have  $\mathcal{M}, w', x \not\models \varphi$ . This contradicts that  $\mathcal{M}, u, x \models \varphi$  for all  $u \in W$ .

*Case 2:* Assume  $\mathcal{M}, w, x \not\models \Box\varphi$ . By the semantic clause for  $\Box$ , there exists  $z \in T$  with  $x < z$  such that  $\mathcal{M}, w, z \not\models \varphi$ . Since  $\mathcal{M}$  is abundant, there exists  $w'' \in W$  such that  $w \approx_z^x w''$ . By **L4**,  $\mathcal{M}, w, z \models \varphi$  just in case  $\mathcal{M}, w'', x \models \varphi$ . Since  $\mathcal{M}, w, z \not\models \varphi$ , we have  $\mathcal{M}, w'', x \not\models \varphi$ . This contradicts that  $\mathcal{M}, u, x \models \varphi$  for all  $u \in W$ .

Both cases yield a contradiction. Therefore  $\mathcal{M}, w, x \models \Box\varphi \rightarrow \Delta\varphi$  for all abundant models  $\mathcal{M}$ , worlds  $w \in W$ , times  $x \in T$ , and well-formed sentences  $\varphi$  of  $\mathcal{L}$ . Hence **P1** is valid over the abundant two-dimensional models of  $\mathcal{L}$ .

Since  $\Box\neg\varphi \rightarrow \Delta\neg\varphi$  by **P1**, contraposition yields  $\neg\Delta\neg\varphi \rightarrow \neg\Box\neg\varphi$ . By definition,  $\nabla\varphi \rightarrow \Diamond\varphi$ , and so **P2** is valid over the abundant two-dimensional models of  $\mathcal{L}$ .  $\square$

**T4** Abundant models with at least two distinct times are unbounded above and below.

*Proof.* Let  $\mathcal{M} = \langle W, T, \leq, |\cdot| \rangle$  be an abundant two-dimensional model of  $\mathcal{L}$  where  $x < y$  for  $x, y \in T$ . We prove that  $T$  is unbounded below and unbounded above.

*Part 1:* Assume for *reductio* that  $T$  is bounded below. Thus there is some  $t_{\min} \in T$  where  $t_{\min} \leq z$  for all  $z \in T$ . Since  $t_{\min} \leq x < y$ , either  $t_{\min} < x$  or  $t_{\min} = x < y$ . In either case, there exists  $t \in \{x, y\}$  with  $t_{\min} < t$ . Letting  $w \in W$ , it follows by abundance that there is some  $w' \in W$  such that  $w \approx_t^{t_{\min}} w'$ . By definition, there is an order automorphism  $\bar{a} : T \rightarrow T$  where  $t = \bar{a}(t_{\min})$ . Since  $t_{\min} \leq z$  for all  $z \in T$  and  $\bar{a}$  is order-preserving, we have  $\bar{a}(t_{\min}) \leq \bar{a}(z)$  for all  $z \in T$ . Since  $\bar{a}$  is surjective, for any  $z' \in T$  there exists  $z \in T$  where  $\bar{a}(z) = z'$ . Thus  $\bar{a}(t_{\min}) \leq z'$  for all  $z' \in T$ , so  $\bar{a}(t_{\min})$  is minimal in  $T$ . Therefore  $\bar{a}(t_{\min}) = t_{\min}$ , and so  $t = t_{\min}$ , which contradicts  $t_{\min} < t$ . We may conclude by *reductio* that  $T$  is unbounded below.

*Part 2:* Assume for *reductio* that  $T$  is bounded above. Thus there is some  $t_{\max} \in T$  where  $z \leq t_{\max}$  for all  $z \in T$ . Since  $x < y \leq t_{\max}$ , either  $y < t_{\max}$  or  $x < y = t_{\max}$ . In either case, there exists  $t \in \{x, y\}$  with  $t < t_{\max}$ . Letting  $w \in W$ , it follows by abundance that there is some  $w' \in W$  such that  $w \approx_t^{t_{\max}} w'$ . By definition, there is an order automorphism  $\bar{a} : T \rightarrow T$  where  $t = \bar{a}(t_{\max})$ . Since  $z \leq t_{\max}$  for all  $z \in T$  and  $\bar{a}$  is order-preserving, we have  $\bar{a}(z) \leq \bar{a}(t_{\max})$  for all  $z \in T$ . Since  $\bar{a}$  is surjective, for any  $z' \in T$  there exists  $z \in T$  where  $\bar{a}(z) = z'$ . Thus  $z' \leq \bar{a}(t_{\max})$  for all  $z' \in T$ , so  $\bar{a}(t_{\max})$  is maximal in  $T$ . Therefore  $\bar{a}(t_{\max}) = t_{\max}$ , and so  $t = t_{\max}$ , which contradicts  $t < t_{\max}$ . We may conclude by *reductio* that  $T$  is unbounded above.  $\square$

## 5.2 Task Semantics

We begin this section by restating a number of definitions for convenience:

**D9** The language  $\mathcal{L} := \langle \mathbb{L}, \perp, \rightarrow, \Box, \Box, \Box \rangle$  where  $\mathbb{L} := \{p_i : i \in \mathbb{N}\}$  is a countable set of sentence letters and the remaining symbols denote falsity, material implication, the metaphysical necessity operator, the universal past tense operator, and the universal future tense operator, respectively.

**D10** A *frame* is a structure  $\mathcal{F} = \langle W, \mathcal{T}, \Rightarrow \rangle$  where:

*World States:* A nonempty set of *world states*  $W$ .

*Temporal Order:* A totally ordered abelian group  $\mathcal{T} = \langle T, +, \leq \rangle$ .

*Task Relation:* A parameterized task relation  $\Rightarrow$  satisfying:

*Nullity:*  $w \Rightarrow_0 w$ .

*Compositionality:* If  $w \Rightarrow_x u$  and  $u \Rightarrow_y v$ , then  $w \Rightarrow_{x+y} v$ .

**D11** A *world history* over a frame  $\mathcal{F} = \langle W, \mathcal{T}, \Rightarrow \rangle$  is a function  $\tau : X \rightarrow W$  where  $X \subseteq T$  is convex and  $\tau(x) \Rightarrow_y \tau(x+y)$  for all times  $x, y \in T$  where both  $x, x+y \in X$ . The set of all world histories over  $\mathcal{F}$  is denoted  $H_{\mathcal{F}}$ .

**D12** A *model* of  $\mathcal{L}$  is a structure  $\mathcal{M} = \langle W, \mathcal{T}, \Rightarrow, |\cdot| \rangle$  where  $\mathcal{F} = \langle W, \mathcal{T}, \Rightarrow \rangle$  is a frame and  $|p_i| \subseteq W$  for every sentence letter  $p_i \in \mathbb{L}$ .

**D13** Truth in a model at a world history and time is defined recursively:

$(p_i)$   $\mathcal{M}, \tau, x \models p_i$  iff  $x \in \text{dom}(\tau)$  and  $\tau(x) \in |p_i|$ .

$(\perp)$   $\mathcal{M}, \tau, x \not\models \perp$ .

$(\rightarrow)$   $\mathcal{M}, \tau, x \models \varphi \rightarrow \psi$  iff  $\mathcal{M}, \tau, x \not\models \varphi$  or  $\mathcal{M}, \tau, x \models \psi$ .

$(\Box)$   $\mathcal{M}, \tau, x \models \Box\varphi$  iff  $\mathcal{M}, \sigma, x \models \varphi$  for all  $\sigma \in H_{\mathcal{F}}$ .

$(\Box)$   $\mathcal{M}, \tau, x \models \Box\varphi$  iff  $\mathcal{M}, \tau, y \models \varphi$  for all  $y \in T$  where  $y < x$ .

$(\Box)$   $\mathcal{M}, \tau, x \models \Box\varphi$  iff  $\mathcal{M}, \tau, y \models \varphi$  for all  $y \in T$  where  $x < y$ .

**D14** Given a frame  $\mathcal{F} = \langle W, \mathcal{T}, \Rightarrow \rangle$ , world histories  $\tau, \sigma \in H_{\mathcal{F}}$  are *time-shifted from  $y$  to  $x$* —written  $\tau \approx_y^x \sigma$ —if and only if there exists an order automorphism  $\bar{a} : T \rightarrow T$  where  $y = \bar{a}(x)$ ,  $\text{dom}(\sigma) = \bar{a}^{-1}(\text{dom}(\tau))$ , and  $\sigma(z) = \tau(\bar{a}(z))$  for all  $z \in \text{dom}(\sigma)$ .

**L5** For any frame  $\mathcal{F} = \langle W, \mathcal{T}, \Rightarrow \rangle$ , world history  $\tau \in H_{\mathcal{F}}$ , and times  $x, y \in T$ , there is a world history  $\sigma \in H_{\mathcal{F}}$  where  $\tau \approx_y^x \sigma$  is witnessed by  $\bar{a}(z) = z - x + y$ .

*Proof.* Let  $\mathcal{F} = \langle W, \mathcal{T}, \Rightarrow \rangle$  be a frame,  $\tau \in H_{\mathcal{F}}$  a world history where  $x, y \in T$  are arbitrary times. Define  $\bar{a} : T \rightarrow T$  by  $\bar{a}(z) = z - x + y$ .

Since  $\mathcal{T} = \langle T, +, \leq \rangle$  is an abelian group,  $\bar{a}$  is a bijection where  $\bar{a}^{-1}(z) = z + x - y$  is the inverse. Supposing  $z_1 \leq z_2$ , it follows that  $z_1 - x + y \leq z_2 - x + y$ , and so  $\bar{a}(z_1) \leq \bar{a}(z_2)$ . Thus  $\bar{a}$  is an order automorphism with  $\bar{a}(x) = y$ .

Letting  $Y = \bar{a}(X)$ , define  $\sigma(z) = \tau(\bar{a}(z))$  for all  $z \in T$  where  $\bar{a}^{-1}(z) \in X$ . Suppose  $z_1, z_2 \in Y$  with  $z_1 \leq z_2$ , so  $\bar{a}^{-1}(z_1), \bar{a}^{-1}(z_2) \in X$  and  $\bar{a}^{-1}(z_1) \leq \bar{a}^{-1}(z_2)$ . Thus  $\bar{a}^{-1}(z) \in X$  by the convexity of  $X$ , and so  $Y$  is convex since  $z = \bar{a}(\bar{a}^{-1}(z)) \in Y$ .

Suppose  $z, z + d \in Y$ . Then  $\bar{a}^{-1}(z), \bar{a}^{-1}(z + d) \in X$  where:

$$\begin{aligned} \sigma(z) \Rightarrow_d \sigma(z + d) &\Leftrightarrow \tau(\bar{a}(z)) \Rightarrow_d \tau(\bar{a}(z + d)) \\ &\Leftrightarrow \tau(\bar{a}(z)) \Rightarrow_d \tau(\bar{a}(z) + d) \end{aligned}$$

The first biconditional is given by the definition of  $\sigma$  and the second biconditional by:

$$\begin{aligned} \bar{a}(z + d) &= (z + d) - x + y \\ &= (z - x + y) + d \\ &= \bar{a}(z) + d. \end{aligned}$$

Since  $\tau$  is a world history, we have  $\tau(\bar{a}(z)) \Rightarrow_d \tau(\bar{a}(z) + d)$ , and so  $\sigma(z) \Rightarrow_d \sigma(z + d)$  as desired. Thus  $\sigma \in H_{\mathcal{F}}$  and  $\tau \approx_y^x \sigma$  is witnessed by  $\bar{a}$ .  $\square$

**L6**  $\mathcal{M}, \tau, y \models \varphi$  just in case  $\mathcal{M}, \sigma, x \models \varphi$  for any model  $\mathcal{M} = \langle W, \mathcal{T}, \Rightarrow, |\cdot| \rangle$  of  $\mathcal{L}$ , well-formed sentence  $\varphi$  of  $\mathcal{L}$ , and world histories  $\tau, \sigma \in H_{\mathcal{F}}$  where  $\tau \approx_y^x \sigma$  is witnessed by the time-shift function  $\bar{a}(z) = z - x + y$ .

*Proof.* We proceed by induction on the complexity of  $\varphi$ .

*Base Case* ( $p_i$ ): The first and third biconditionals follow from **D13**.

$$\begin{aligned} \mathcal{M}, \tau, y \models p_i &\Leftrightarrow y \in \text{dom}(\tau) \text{ and } \tau(y) \in |p_i| \\ &\Leftrightarrow x \in \text{dom}(\sigma) \text{ and } \sigma(x) \in |p_i| \\ &\Leftrightarrow \mathcal{M}, \sigma, x \models p_i \end{aligned}$$

For the second biconditional, we have  $\text{dom}(\sigma) = \bar{a}^{-1}(\text{dom}(\tau))$  by **D14**, so  $x \in \text{dom}(\sigma)$  just in case  $\bar{a}(x) = y \in \text{dom}(\tau)$ . When both  $x \in \text{dom}(\sigma)$  and  $y \in \text{dom}(\tau)$  it follows that  $\sigma(x) = \tau(\bar{a}(x)) = \tau(y)$ , and so the second biconditional holds. Otherwise, second biconditional holds since both sides are false.

*Base Case* ( $\perp$ ): By **D13**,  $\mathcal{M}, \tau, y \not\models \perp$  and  $\mathcal{M}, \sigma, x \not\models \perp$ .

*Inductive Case* ( $\varphi \rightarrow \psi$ ): The first and third biconditionals follow from **D13** and the second biconditional follows from the inductive hypothesis.

$$\begin{aligned} \mathcal{M}, \tau, y \models \varphi \rightarrow \psi &\Leftrightarrow \mathcal{M}, \tau, y \not\models \varphi \text{ or } \mathcal{M}, \tau, y \models \psi \\ &\Leftrightarrow \mathcal{M}, \sigma, x \not\models \varphi \text{ or } \mathcal{M}, \sigma, x \models \psi \\ &\Leftrightarrow \mathcal{M}, \sigma, x \models \varphi \rightarrow \psi \end{aligned}$$

*Inductive Case* ( $\Box\varphi$ ): The first and third biconditionals follow from **D13**.

$$\begin{aligned}\mathcal{M}, \tau, y \not\models \Box\varphi &\Leftrightarrow \mathcal{M}, \rho, y \not\models \varphi \text{ for some } \rho \in H_{\mathcal{F}} \\ &\Leftrightarrow \mathcal{M}, \rho', x \not\models \varphi \text{ for some } \rho' \in H_{\mathcal{F}} \\ &\Leftrightarrow \mathcal{M}, \sigma, x \not\models \Box\varphi\end{aligned}$$

To justify the forward direction of the second biconditional, assume  $\rho \in H_{\mathcal{F}}$  with  $\mathcal{M}, \rho, y \not\models \varphi$ . By **L5** there is  $\rho' \in H_{\mathcal{F}}$  where  $\rho \approx_y^x \rho'$  is witnessed by  $\bar{a}(z) = z - x + y$ , so  $\mathcal{M}, \rho', x \not\models \varphi$  by the inductive hypothesis. Now assume  $\rho' \in H_{\mathcal{F}}$  with  $\mathcal{M}, \rho', x \not\models \varphi$ . By **L5**, there is  $\rho \in H_{\mathcal{F}}$  where  $\rho' \approx_x^y \rho$  is witnessed by  $\bar{a}^{-1}(z) = z - y + x$ , so by the inductive hypothesis  $\mathcal{M}, \rho, y \not\models \varphi$ .

*Inductive Case* ( $\Box\varphi$ ): The first and third biconditionals follow from **D13**.

$$\begin{aligned}\mathcal{M}, \tau, y \not\models \Box\varphi &\Leftrightarrow \mathcal{M}, \tau, y' \not\models \varphi \text{ for some } y' \in T \text{ where } y' < y \\ &\Leftrightarrow \mathcal{M}, \sigma, x' \not\models \varphi \text{ for some } x' \in T \text{ where } x' < x \\ &\Leftrightarrow \mathcal{M}, \sigma, x \not\models \Box\varphi\end{aligned}$$

To justify the forward direction of the second biconditional, assume  $\mathcal{M}, \tau, y' \not\models \varphi$  for some  $y' < y$ . Letting  $x' = \bar{a}^{-1}(y') = y' + x - y$ , we know that  $y' = \bar{a}(x')$ . Since  $y = \bar{a}(x)$ , we have  $\bar{a}(x') < \bar{a}(x)$ . Thus  $x' < x$  since  $\bar{a}$  is an order automorphism. By assumption  $\tau \approx_y^x \sigma$ , and so  $\text{dom}(\sigma) = \bar{a}^{-1}(\text{dom}(\tau))$  and  $\sigma(z) = \tau(\bar{a}(z))$  for all  $z \in \text{dom}(\sigma)$  by **D14**. Given that  $\bar{a}(x') = y'$ , the same  $\bar{a}$  that witnesses  $\tau \approx_y^x \sigma$  also witnesses  $\tau \approx_{y'}^{x'} \sigma$ . It follows by the inductive hypothesis that  $\mathcal{M}, \sigma, x' \not\models \varphi$ .

Now assume  $\mathcal{M}, \sigma, x' \not\models \varphi$  for some  $x' < x$ , letting  $y' = \bar{a}(x') = x' - x + y$ . Since  $\bar{a}$  is an order automorphism, it follows that  $\bar{a}(x') < \bar{a}(x)$ , and so  $y' < y$  given that  $\bar{a}(x) = y$ . We know that  $\bar{a}(x') = y'$ , and so it follows by the same reasoning above that  $\bar{a}$  witnesses  $\tau \approx_{y'}^{x'} \sigma$ , and so by the inductive hypothesis  $\mathcal{M}, \tau, y' \not\models \varphi$ .

*Inductive Case* ( $\Box\varphi$ ): The proof is similar to the  $\Box$  case.  $\square$

**T5** Given the semantics for  $\mathcal{L}$ , both **P1** and **P2** are valid.

*Proof.* Suppose for contradiction that there is a model  $\mathcal{M} = \langle W, \mathcal{T}, \Rightarrow, |\cdot| \rangle$ , a world history  $\tau \in H_{\mathcal{F}}$ , and a time  $x \in \mathcal{T}$  such that  $\mathcal{M}, \tau, x \not\models \Box\varphi \rightarrow \Delta\varphi$ , and so  $\mathcal{M}, \tau, x \models \Box\varphi$  but  $\mathcal{M}, \tau, x \not\models \Delta\varphi$ . Since  $\Delta\varphi := \Box\varphi \wedge \varphi \wedge \Box\varphi$ , we have  $\mathcal{M}, \tau, x \not\models \Box\varphi$  or  $\mathcal{M}, \tau, x \not\models \varphi$  or  $\mathcal{M}, \tau, x \not\models \Box\varphi$ . However, from  $\mathcal{M}, \tau, x \models \Box\varphi$ , we know that  $\mathcal{M}, \sigma, x \models \varphi$  for all world histories  $\sigma \in H_{\mathcal{F}}$ , and so  $\mathcal{M}, \tau, x \models \varphi$ . Thus either  $\mathcal{M}, \tau, x \not\models \Box\varphi$  or  $\mathcal{M}, \tau, x \not\models \Box\varphi$ , and so  $\mathcal{M}, \tau, y \not\models \varphi$  for some time  $y \in T$  where either  $y < x$  or  $y > x$ .

*Case 1* ( $y < x$ ): By **L5**, there is a world history  $\sigma \in H_{\mathcal{F}}$  where  $\tau \approx_y^x \sigma$  is witnessed by  $\bar{a}(z) = z - x + y$ . By **L6**,  $\mathcal{M}, \tau, y \models \varphi$  just in case  $\mathcal{M}, \sigma, x \models \varphi$ . Since  $\mathcal{M}, \tau, y \not\models \varphi$ , it follows that  $\mathcal{M}, \sigma, x \not\models \varphi$ , and so  $\mathcal{M}, \tau, x \not\models \Box\varphi$ , contradicting the above.

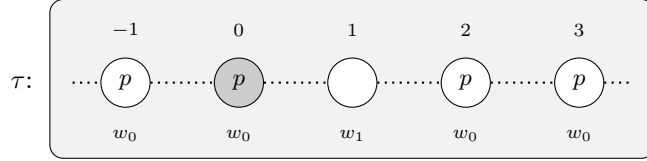
*Case 2* ( $y > x$ ): The proof is symmetric to *Case 1*.

Therefore,  $\models \Box\varphi \rightarrow \Delta\varphi$ . It follows that  $\models \Box\neg\varphi \rightarrow \Delta\neg\varphi$ , and so  $\models \nabla\varphi \rightarrow \Diamond\varphi$  by contraposition and the duality of the modals. This proves **P1** and **P2** are valid.  $\square$

**T6**  $\boxed{\text{F}}\boxed{\text{F}}\varphi \rightarrow \boxed{\text{F}}\varphi$  is invalid over all models of  $\mathcal{L}$ .

*Proof.* Let  $\mathcal{F} = \langle W, \mathcal{T}, \Rightarrow \rangle$  be a frame with world states  $W = \{w_0, w_1\}$ , temporal order  $\mathcal{T} = \langle \mathbb{Z}, +, \leq \rangle$  where  $\mathbb{Z}$  is the integers with standard addition  $+$  and order  $\leq$ , and task relation  $w \Rightarrow_n w'$  for all  $w, w' \in W$  and  $n \in \mathbb{Z}$ . This frame satisfies NULLITY and COMPOSITIONALITY by definition.

Let  $\mathcal{M} = \langle W, \mathcal{T}, \Rightarrow, |\cdot| \rangle$  be a model over  $\mathcal{F}$  with  $|p| = \{w_0\}$ . Define world history  $\tau : \mathbb{Z} \rightarrow W$  by  $\tau(1) = w_1$  and  $\tau(n) = w_0$  for all  $n \neq 1$ . Since  $\text{dom}(\tau) = \mathbb{Z}$  is convex and  $\tau(x) \Rightarrow_y \tau(x+y)$  holds for all  $x, y \in \mathbb{Z}$  by construction, we have  $\tau \in H_{\mathcal{F}}$  by **D11**.



Since  $1 > 0$  and  $\tau(1) = w_1 \notin |p|$ , we have  $1 \in \text{dom}(\tau)$  and  $\mathcal{M}, \tau, 1 \not\models p$  by **D13**. Therefore  $\mathcal{M}, \tau, 0 \not\models \boxed{\text{F}}p$ . Letting  $y > 0$  and  $z > y$ , we have  $z \geq 2$ . Since  $z \neq 1$ , we have  $\tau(z) = w_0 \in |p|$ , and so  $\mathcal{M}, \tau, z \models p$  by **D13**. Thus  $\mathcal{M}, \tau, y \models \boxed{\text{F}}p$  for all  $y > 0$ , and so  $\mathcal{M}, \tau, 0 \models \boxed{\text{F}}\boxed{\text{F}}p$ . It follows that  $\mathcal{M}, \tau, 0 \not\models \boxed{\text{F}}\boxed{\text{F}}p \rightarrow \boxed{\text{F}}p$ .  $\square$

**T7**  $\boxed{\text{F}}\boxed{\text{F}}\varphi \rightarrow \boxed{\text{F}}\varphi$  is valid over the DENSE models of  $\mathcal{L}$ .

*Proof.* Let  $\mathcal{M} = \langle W, \mathcal{T}, \Rightarrow, |\cdot| \rangle$  be a DENSE model of  $\mathcal{L}$  and assume for contradiction that  $\mathcal{M}, \tau, x \not\models \boxed{\text{F}}\boxed{\text{F}}\varphi \rightarrow \boxed{\text{F}}\varphi$  for a world history  $\tau \in H_{\mathcal{F}}$ , and a time  $x \in T$ . Then  $\mathcal{M}, \tau, x \models \boxed{\text{F}}\boxed{\text{F}}\varphi$  and  $\mathcal{M}, \tau, x \not\models \boxed{\text{F}}\varphi$ . It follows that there is some  $y \in T$  where  $y > x$  and  $\mathcal{M}, \tau, y \not\models \varphi$  by **D13**. Since  $\mathcal{T}$  is dense and  $x < y$ , there is a  $z \in T$  where  $x < z < y$ . Given that  $\mathcal{M}, \tau, x \models \boxed{\text{F}}\boxed{\text{F}}\varphi$  and  $z > x$ , we have  $\mathcal{M}, \tau, z \models \boxed{\text{F}}\varphi$  by **D13**. Thus for all  $z' \in T$  where  $z' > z$ , we have  $\mathcal{M}, \tau, z' \models \varphi$  by **D13**. Since  $y > z$ , it follows that  $\mathcal{M}, \tau, y \models \varphi$ , contradicting  $\mathcal{M}, \tau, y \not\models \varphi$  above. Therefore  $\boxed{\text{F}}\boxed{\text{F}}\varphi \rightarrow \boxed{\text{F}}\varphi$  is valid over all DENSE models of  $\mathcal{L}$  as desired.  $\square$

### 5.3 Proof Theory

Since derivations of **P1** – **P6** are provided in §3.2 and **P7** – **P8** concern an extension of **TM** that includes **DN**, this section will derive a number of additional interaction principles in **TM**. I will begin by stating the following equivalences without proof.

$$\mathbf{P9} \quad \neg\Delta\varphi \leftrightarrow \nabla\neg\varphi.$$

$$\mathbf{P10} \quad \neg\nabla\varphi \leftrightarrow \Delta\neg\varphi.$$

$$\mathbf{P11} \quad \nabla\Diamond\varphi \rightarrow \Diamond\varphi.$$

*Proof.* Since  $\nabla\Diamond\varphi := \Diamond\Diamond\varphi \vee \Diamond\varphi \vee \Diamond\Diamond\varphi$  where  $\Diamond\varphi \rightarrow \Diamond\varphi$  is a theorem of classical propositional logic, it suffices to show  $\Diamond\Diamond\varphi \rightarrow \Diamond\varphi$  and  $\Diamond\Diamond\varphi \rightarrow \Diamond\varphi$ . By **TF** instantiated

with  $\neg\varphi$ , we have  $\Box\neg\varphi \rightarrow \boxed{\text{F}}\Box\neg\varphi$ , so  $\Diamond\Diamond\varphi \rightarrow \Diamond\varphi$  by contraposition. By **TD** applied to **TF**, we have  $\Box\neg\varphi \rightarrow \boxed{\text{P}}\Box\neg\varphi$ , and by contraposition,  $\Diamond\Diamond\varphi \rightarrow \Diamond\varphi$ .  $\square$

**P12**  $\Delta\Diamond\varphi \rightarrow \Diamond\varphi$ .

*Proof.* Since  $\Delta\Diamond\varphi := \boxed{\text{F}}\Diamond\varphi \wedge \Diamond\varphi \wedge \boxed{\text{F}}\Diamond\varphi$ , conjunction elimination yields  $\Delta\Diamond\varphi \rightarrow \Diamond\varphi$ .  $\square$

**P13**  $\nabla\Box\varphi \leftrightarrow \Box\varphi$ .

*Proof.* By **P6**, we have  $\nabla\Box\varphi \rightarrow \Box\Delta\varphi$ . Since  $\Delta\varphi \rightarrow \varphi$  by definition and propositional logic, **MK** yields  $\Box\Delta\varphi \rightarrow \Box\varphi$ . Therefore  $\nabla\Box\varphi \rightarrow \Box\varphi$ . Since  $\nabla\varphi := \Diamond\varphi \vee \varphi \vee \Diamond\varphi$ , the reverse direction  $\Box\varphi \rightarrow \nabla\Box\varphi$  follows by disjunction introduction.  $\square$

**P14**  $\Delta\Box\varphi \leftrightarrow \Box\varphi$ .

*Proof.* Given the definition  $\Delta\varphi := \boxed{\text{P}}\varphi \wedge \varphi \wedge \boxed{\text{F}}\varphi$ , the forward direction  $\Delta\Box\varphi \rightarrow \Box\varphi$  follows by conjunction elimination. For the reverse direction, we have  $\Box\varphi \rightarrow \Box\varphi$  trivially,  $\Box\varphi \rightarrow \boxed{\text{F}}\Box\varphi$  by **TF**, and  $\Box\varphi \rightarrow \boxed{\text{P}}\Box\varphi$  by **TD** applied to **TF**. Therefore  $\Box\varphi \rightarrow (\boxed{\text{P}}\Box\varphi \wedge \Box\varphi \wedge \boxed{\text{F}}\Box\varphi)$ , which is  $\Box\varphi \rightarrow \Delta\Box\varphi$  as desired.  $\square$

**P15**  $\Box\Delta\varphi \leftrightarrow \Delta\Box\varphi$ .

*Proof.* Since  $\Delta\varphi \rightarrow \varphi$  by definition, **MK** yields  $\Box\Delta\varphi \rightarrow \Box\varphi$ . By **P14**,  $\Box\varphi \rightarrow \Delta\Box\varphi$ , and so  $\Box\Delta\varphi \rightarrow \Delta\Box\varphi$ , establishing the forward direction. Since **P14** gives  $\Delta\Box\varphi \rightarrow \Box\varphi$ , we know  $\Box\varphi \rightarrow \Box\Delta\varphi$  by **P3**, and so  $\Delta\Box\varphi \rightarrow \Box\Delta\varphi$ , establishing the reverse.  $\square$

**P16**  $\Diamond\nabla\varphi \leftrightarrow \nabla\Diamond\varphi$ .

*Proof.* By **P10**,  $\neg\nabla\varphi \leftrightarrow \Delta\neg\varphi$ . From **MK** we may obtain  $\Box\neg\nabla\varphi \leftrightarrow \Box\Delta\neg\varphi$ , and so  $\Diamond\nabla\varphi \leftrightarrow \neg\Box\Delta\neg\varphi$ . Instantiating **P15** with  $\neg\varphi$  yields  $\Box\Delta\neg\varphi \leftrightarrow \Delta\Box\neg\varphi$ , and so we get  $\neg\Box\Delta\neg\varphi \leftrightarrow \neg\Delta\Box\neg\varphi$ . By instantiating **P9** with  $\Box\neg\varphi$ , we have  $\neg\Delta\Box\neg\varphi \leftrightarrow \nabla\neg\Box\neg\varphi$ , and so  $\neg\Delta\Box\neg\varphi \leftrightarrow \nabla\Diamond\varphi$ . Thus  $\Diamond\nabla\varphi \leftrightarrow \nabla\Diamond\varphi$  by the transitivity of biconditionals.  $\square$

**P17**  $\Diamond\nabla\varphi \rightarrow \Diamond\varphi$ .

*Proof.* Follows immediately from **P11** and **P16**.  $\square$

**P18**  $\Box\Delta\varphi \leftrightarrow \Box\varphi$ .

*Proof.* For the forward direction, since  $\Delta\varphi \rightarrow \varphi$  by definition and propositional logic, **MK** yields  $\Box\Delta\varphi \rightarrow \Box\varphi$ . For the reverse direction, by **P1**, we have  $\Box\varphi \rightarrow \Delta\varphi$ , so **MK** yields  $\Box\Box\varphi \rightarrow \Box\Delta\varphi$ . Since  $\Box\varphi \rightarrow \Box\Box\varphi$  by **M4**, we obtain  $\Box\varphi \rightarrow \Box\Delta\varphi$  as desired.  $\square$

**P19**  $\Diamond\Delta\varphi \rightarrow \Diamond\varphi$ .

*Proof.* Since  $\Delta\varphi \rightarrow \varphi$  by propositional logic, we get  $\neg\varphi \rightarrow \neg\Delta\varphi$  by contraposition. By **MK**, we obtain  $\Box\neg\varphi \rightarrow \Box\neg\Delta\varphi$ , and so  $\Diamond\Delta\varphi \rightarrow \Diamond\varphi$  by contraposition.  $\square$

**P20**  $\Diamond\varphi \leftrightarrow \Diamond\nabla\varphi$ .

*Proof.* Instantiating **P18** with  $\neg\varphi$  yields  $\Box\Delta\neg\varphi \leftrightarrow \Box\neg\varphi$ . By **P10**,  $\neg\nabla\varphi \leftrightarrow \Delta\neg\varphi$ , so from **MK** we obtain  $\Box\neg\nabla\varphi \leftrightarrow \Box\Delta\neg\varphi$ . Therefore  $\Box\neg\nabla\varphi \leftrightarrow \Box\neg\varphi$  by transitivity of biconditionals, and taking negations of both sides gives  $\Diamond\nabla\varphi \leftrightarrow \Diamond\varphi$  as desired.  $\square$

**P21**  $\nabla\Diamond\varphi \rightarrow \Diamond\nabla\varphi$ .

*Proof.* Follows immediately from **P11** and **P20**.  $\square$

**P22**  $\Diamond\nabla\varphi \rightarrow \nabla\Diamond\varphi$ .

*Proof.* By **P5**, we have  $\Diamond\nabla\varphi \rightarrow \Delta\Diamond\varphi$ . Since  $\Delta\Diamond\varphi \rightarrow \Diamond\varphi$  by conjunction elimination, and  $\Diamond\varphi \rightarrow \nabla\Diamond\varphi$  by definition, we obtain  $\Diamond\nabla\varphi \rightarrow \nabla\Diamond\varphi$  as desired.  $\square$

## References

- [1] Montague, R.: The Proper Treatment of Quantification in Ordinary English. In: Hintikka, K.J.J., Moravcsik, J.M.E., Suppes, P. (eds.) *Approaches to Natural Language: Proceedings of the 1970 Stanford Workshop on Grammar and Semantics*, pp. 221–242. Springer, Dordrecht (1973). [https://doi.org/10.1007/978-94-010-2506-5\\_10](https://doi.org/10.1007/978-94-010-2506-5_10)
- [2] Kaplan, D.: Demonstratives: An Essay on the Semantics, Logic, Metaphysics and Epistemology of Demonstratives and Other Indexicals. In: Almog, J., Perry, J., Wettstein, H. (eds.) *Themes From Kaplan*, pp. 481–563. Oxford University Press, New York (1989)
- [3] Lewis, C.I., Langford, C.H.: *Symbolic Logic*. Century Company, New York (1932)
- [4] Kripke, S.A.: Semantical Analysis of Modal Logic I Normal Modal Propositional Calculi. *Mathematical Logic Quarterly* **9**(5-6), 67–96 (1963) <https://doi.org/10.1002/malq.19630090502>
- [5] Kripke, S.A.: Semantical Considerations on Modal Logic. *Acta Philosophica Fennica* **16**(1963), 83–94 (1963)
- [6] Carnap, R.: *Modalities And Quantification*. The Journal of Symbolic Logic (1946)
- [7] Carnap, R.: *Meaning and Necessity*. University of Chicago Press, Chicago (1947)
- [8] Carnap, R.: *Introduction to Semantics*. Harvard University Press, Cambridge, MA (1942)
- [9] Kripke, S.A.: A completeness theorem in modal logic. *The Journal of Symbolic Logic* **24**(1), 1–14 (1959) <https://doi.org/10.2307/2964568>
- [10] Kripke, S.A.: Identity and Necessity. In: Munitz, M.K. (ed.) *Identity and*

- Individuation, pp. 135–164. New York University Press, New York (1971)
- [11] Kripke, S.A.: Naming and Necessity. In: *Semantics of Natural Language*, pp. 253–355. Springer, Dordrecht (1972). [https://doi.org/10.1007/978-94-010-2557-7\\_9](https://doi.org/10.1007/978-94-010-2557-7_9)
  - [12] Kripke, S.A.: Letter from Saul Kripke to A.N. Prior, Oct. 13, 1958 (1958)
  - [13] Prior, A.N.: *Time and Modality*. Greenwood Press, Westport, Conn. (1955)
  - [14] Ploug, T., Øhrstrøm, P.: Branching time, indeterminism and tense logic. *Synthese* **188**(3), 367–379 (2012) <https://doi.org/10.1007/s11229-011-9944-2>
  - [15] Dorr, C., Goodman, J.: Diamonds are Forever. *Noûs* **54**(3), 632–665 (2020) <https://doi.org/10.1111/nous.12271>
  - [16] Prior, A.N.: *Past, Present and Future*. Oxford University Press, Oxford, New York (1967)
  - [17] Thomason, R.H.: Indeterminist time and truth-value gaps. *Theoria* **36**(3), 264–281 (1970) <https://doi.org/10.1111/j.1755-2567.1970.tb00427.x>
  - [18] Montague, R.: *Formal Philosophy: Selected Papers*. Yale University Press, New Haven (1974)
  - [19] Fine, K.: Prior on the construction of possible worlds and instants. *World, times, and selves* (1977)
  - [20] Kripke, S.A.: Letter from Saul Kripke to A. N. Prior, Sept. 3, 1958 (1958)
  - [21] Broad, C.D.: *Examination of McTaggart’s Philosophy vol. II*. Cambridge University Press, London (1938)
  - [22] McTaggart, J.E.: The Unreality of Time. *Mind* **XVII**(4), 457–474 (1908) <https://doi.org/10.1093/mind/XVII.4.457>
  - [23] Reichenbach, H.: *Elements of Symbolic Logic*. Macmillan Co., New York, (1947)
  - [24] Müller, T.: Alternatives to Histories? Employing a Local Notion of Modal Consistency in Branching Theories. *Erkenntnis* **79**(3), 343–364 (2014) <https://doi.org/10.1007/s10670-013-9453-4>
  - [25] Rumberg, A.: Transition Semantics for Branching Time. *Journal of Logic, Language and Information* **25**(1), 77–108 (2016) <https://doi.org/10.1007/s10849-015-9231-6>
  - [26] Brast-McKie, B.: Counterfactual Worlds. *Journal of Philosophical Logic*, 1–42 (2025) <https://doi.org/10.1007/s10992-025-09793-8>

- [27] Salmon, N.: The Logic of What Might Have Been. *The Philosophical Review* **98**(1), 3–34 (1989) <https://doi.org/10.2307/2185369>
- [28] Kamp, H.: Tense Logic and the Theory of Linear Order. PhD thesis, Ucla (1968)
- [29] Williamson, T.: Modal science. *Canadian Journal of Philosophy* **46**(4-5), 453–492 (2016) <https://doi.org/10.1080/00455091.2016.1205851>
- [30] Thomason, R.H.: Combinations of Tense and Modality. In: Gabbay, D., Guentner, F. (eds.) *Handbook of Philosophical Logic: Volume II: Extensions of Classical Logic*. Synthese Library, pp. 135–165. Springer, Dordrecht (1984). <https://doi.org/10.1007/978-94-009-6259-0-3>