The Construction of Possible Worlds

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Abstract

This paper presents the *Logic for Tense and Modality* (**TM**), a sound logic for non-deterministic dynamical systems whose discrete, dense, and continuous $\mathbf{TM}^+$ extensions are completely axiomatized, providing resources for reasoning about past and future contingency. By taking *world states* to be total configuration states for a system, the *task relation* encodes the possible transitions between world states over a *duration*. Given a model of the bimodal language, sentences are evaluated at a *possible world* and *time* where each possible world is a function from durations to world states constrained by the task relation. By contrast with two-dimensional semantic theories which take both times and worlds to be primitive, the perpetuity principle that what is sometimes the case is possible is a theorem in **TM**. The paper begins by reviewing the origins of tense and modal logic in §**2** to motivate the construction of possible worlds in §**3** and resulting theory of tense and modality in §**4**. The *Appendix* provides supporting formal results in §**5**. Section §**1** begins by characterizing tense and modality by way of the perpetuity principles which §**5** derives as theorems of **TM**.

Keywords: Dynamical Systems, Tense, Modality, Bimodal Logic, Task Semantics

1 Introduction

Intensional semantic theories often conceive of possible worlds as complete histories of everything. Insofar as there are *temporary sentences* which are true at some times and false at others in the same possible world, the truth-value of a temporary sentence is not fixed by a possible world considered on its own. For instance, suppose that in a world w, the sentence 'Cary is reading' was true this morning but false in the afternoon. Merely specifying the world of evaluation w does not determine the truth-value of a temporary sentence, at least insofar as possible worlds are taken to be temporally extended histories rather than instantaneous world states.

I am grateful to Miguel Buitrago, Bailey Fernandez, Johan van Benthem, Jeremy Goodman, Graham Leach-Krouse, Dana Scott, Jonas Werner, Steven Yablo, the attendees of the seminar I taught at MIT (Spring 2025) on *The Modern History of Modal Logic*. I am especially grateful to Steve Kuhn for detailed comments and to Tim Williamson for inspiring this paper and deeply influencing my views on modality.

One response denies that there are temporary sentences, assuming instead that every sentence, perhaps implicitly, includes a reference to some time or other. An *eternalist* of this kind takes the sentence ‘Cary is reading’ to be incomplete and so to be replaced by the *permanent sentence* ‘Cary is reading at t’ where t is a time.[1] However, including singular terms which refer to *times* makes an ontology of times a part of the topic of conversation. Although some sentences may be about certain times either imagined, fictional, experienced, theorized about, or otherwise discussed, the sentence ‘Cary is reading’ does not concern any time whatsoever, but rather has Cary and her engagement reading as the entire subject-matter. Moreover, since a permanent sentence that is true in w is true at any time in w, a permanent sentence φ is equivalent to the result of embedding φ under arbitrary tense operators, obviating the need to include tense operators in the language.[2] However, the truth-condition for the sentence ‘Cary is reading’ differs substantially from the result of embedding this sentence under tense operators. For instance, it might be true that Cary is reading, and false that she always has been reading, or is always going to be reading. Instead of excluding tense operators and temporary sentences such as ‘Cary is reading’ from the language, this paper provides a truth-conditional semantics which respects the natural morphology of tensed and modal claims, describing their interactions in a bimodal language without positing unspoken references to an ontology of times.

Rather than including singular terms for times in the logical form of the sentences of a language, the semantics pioneered by Montague [1] and Kaplan [2] evaluates sentences at world-time pairs. Assigning sentences to sets of world-time pairs makes times endogenous to the interpretation of the language.[3] This strategy provides a two-dimensional semantics for $\boxed{\scriptstyle\mathrm{P}}$ and $\boxed{\scriptstyle\mathrm{F}}$ which read ‘It has always been the case that’ and ‘It is always going to be the case that’, respectively. By evaluating sentences at world-time pairs, it is easy to distinguish the truth-condition for ‘Cary is reading’ from the result of embedding this sentence under tense operators which shift the time of evaluation while holding the world of evaluation fixed. Whereas ‘Cary is reading’ is assigned to the set of all world-time pairs in which Cary is reading in that world at that time, ‘Cary always was reading’ is assigned to the set of all world-time pairs where she is reading at all earlier times in that world.

Although two-dimensional semantics admits distinct truth-conditions for sentences of the form φ, $\boxed{\scriptstyle\mathrm{P}}\varphi$, $\boxed{\scriptstyle\mathrm{F}}\varphi$, $\Diamond_{\mathrm{P}}\varphi$, $\Diamond_{\mathrm{F}}\varphi$ and sentences with iterated temporal operators, taking possible worlds to be structureless points within a model while nevertheless representing temporally extended histories comes at both an intuitive and theoretical cost. Letting $\Diamond_{\mathrm{P}}\varphi \coloneqq \neg\boxed{\scriptstyle\mathrm{P}}\neg\varphi$ and $\Diamond_{\mathrm{F}}\varphi \coloneqq \neg\boxed{\scriptstyle\mathrm{F}}\neg\varphi$ express ⌜It has been the case that φ⌝ and ⌜It is going to be the case that φ⌝, I will take $\triangle\varphi \coloneqq \boxed{\scriptstyle\mathrm{P}}\varphi \wedge \varphi \wedge \boxed{\scriptstyle\mathrm{F}}\varphi$ and $\triangledown\varphi \coloneqq \Diamond_{\mathrm{P}}\varphi \vee \varphi \vee \Diamond_{\mathrm{F}}\varphi$ to read ⌜It is always the case that φ⌝ and ⌜It is sometimes the case that φ⌝ where $\Box$ and $\Diamond$ are the metaphysical modals which shift the world of evaluation while holding the time fixed. A semantics of this kind makes the following *perpetuity principles* invalid:

[1] Atomic permanent sentences could be regimented by ‘$R_t(c)$’ or ‘$R(c,t)$’. What matters is that the time t is included in the logical form rather than as a parameter at which the sentence is evaluated.

[2] There are pathological cases if time is bounded, though these exceptions are beside the present point.

[3] Alternatively, sentences could be assigned to characteristic functions from world-time pairs to truth-values, or else functions from times (perhaps together with other contextual parameters) to functions from worlds to truth-values, all of which take times to be endogenous to the interpretation of the language.

P1 $\Box\varphi \to \triangle\varphi$. **P2** $\triangledown\varphi \to \Diamond\varphi$.

So long as sentence letters may be assigned to any set of world-time pairs where there is more than one time, φ may be true in every world w' at a time x without also being true in a world w at every time x'. Insofar as $\Box\varphi$ is true in a world w at time x just in case φ is true in u at time x for every world u, it follows that **P1** is invalid where **P2** is also invalid on account of being logically equivalent. However, it is natural to assume that whatever is metaphysically necessary is always the case, or equivalently, whatever is sometimes the case is metaphysically possible.

To bring out the metaphysical reading of the modal operators $\Box$ and $\Diamond$ that I will assume, consider Alvin who recently turned ten years old $\Diamond\!\!\!\!\triangledown\texttt{Ten}$. Although it follows that there is a time where Alvin turned ten $\triangledown\texttt{Ten}$, one might deny that it is *now* possible for Alvin to turn ten $\neg\Diamond\texttt{Ten}$, and so **P2** is false. It may then be taken to be an advantage to make **P1** and **P2** invalid to accommodate such examples. This example highlights that $\Box$ and $\Diamond$ cannot be assimilated to the natural language expressions 'necessarily' and 'possibly' which quantify over a contextually restricted set of possible worlds rather than all objective possible worlds. For instance, it is often natural to restrict consideration to some relevant subset of possible worlds that overlap with the past leading up to the present. By contrast, the metaphysical modals quantify over all possible worlds, including those that bear no relation to the actual past from our present perspective. In the case of Alvin turning ten years old, we may consider possible worlds just like the world imagined to be actual, in which Alvin turned ten years old some time ago but where events are shifted forward such that he is just now turning ten years old. Insofar as the metaphysical modal operators quantify over all such possible worlds, it follows immediately that it *is* possible for Alvin to turn ten years old, though not in any possible world that shares our actual past.

I will take metaphysical modality to be the strongest objective modality.[4] Letting $\ulcorner\varphi \equiv \psi\urcorner$ read $\ulcorner$For it to be the case that φ just is for it to be the case that $\psi\urcorner$ where $\ulcorner\vec{\chi}_{(\psi/\varphi)}\urcorner$ is the result of freely substituting ψ for φ among the arguments $\vec{\chi}$, an n-place operator $\mathcal{Q}$ is *objective* only if all instances of the following schema hold:

Transparency: $(\varphi \equiv \psi) \to [\mathcal{Q}(\vec{\chi}) \equiv \mathcal{Q}(\vec{\chi}_{(\psi/\varphi)})]$.[5]

Letting $\boxed{\scriptstyle N}$ read 'It is nomologically necessary that', I will assume that $\boxed{\scriptstyle N}$ is also an objective modal operator, but strictly weaker than the metaphysical modals. Rather than quantifying over all possible worlds, $\boxed{\scriptstyle N}\varphi$ is true in a world w at time x just in case φ is true in u at x for every world u which conforms to the laws of nature in w at x. Whereas $\Box\varphi \to \boxed{\scriptstyle N}\varphi$ is valid, $\boxed{\scriptstyle N}\varphi \to \Box\varphi$ admits counterexamples insofar as there are metaphysically possible worlds that do not obey the laws of nature. In contrast to the metaphysical and nomological modals, the operator 'Homer believes that' is not transparent, and so is not objective. For instance, 'Homer believes Hesperus is rising' and 'Homer believes that Phosphorus is rising' may differ in truth-value despite the

[4]See Williamson [3–5] for discussion. The formal characterization of $\Box$ as the strongest objective normal modal operator is developed in **§5.1**, where an axiomatic theory of the objective modalities establishes existence in **T2** and uniqueness up to equivalence in **L2**, with a coarse-grained comparison drawn in **C1**.

[5]This transparency schema states a necessary condition on objectivity. See **§5.1** in the *Appendix* for the quantified transparency condition relative to a logic, where the definition of $\mathrm{Obj}(\mathcal{Q})$ builds in factivity.

fact that Hesperus is Phosphorus and so 'Hesperus is rising' and 'Phosphorus is rising' express the same proposition, though this was unknown to Homer.

This paper presents an intensional *task semantics* for the bimodal language $\mathcal{L}$ which includes tense and modal operators for reasoning about non-deterministic dynamical systems consisting of total configuration states.[6] Instead of taking both possible worlds and times to be primitive as assumed by two-dimensional semantic theories, primitive *world states*, *tasks*, and *durations* are introduced in **§3** to define possible worlds as functions from durations to world states appropriately constrained by the task relation. Given a possible world τ, each duration x fixes a world state that occurs x duration from the origin in τ. Each world-time pair $\langle \tau, x \rangle$ is a *moment* where x is referred to as a *time* in τ. By evaluating sentences at moments where the semantics for $\Box$ and $\Diamond$ quantify over all possible worlds and the semantics for $\boxed{\scriptstyle P}$ and $\boxed{\scriptstyle F}$ quantify over past and future times respectively, the perpetuity principles are shown to be valid. After presenting the sound *Logic of Tense and Modality* (**TM**) and its extensions, along with the complete systems for $\mathcal{L}^+$ in the appendix, **§4** provides an account of the openness of the future before drawing further connections to dynamical systems theory and adjacent logics in computer science.[7] The formal results referred to throughout will be provided by the *Appendix* in **§5**. The following section will begin by motivating the construction of possible worlds by reviewing the history of tense and modal logic.

2 Primitive Worlds

In order to provide a flexible semantics with which to characterize the modal systems that Lewis and Langford [7] first set out in Appendix II of their 1932 textbook, Kripke [8, 9] took the models of a modal language to include a nonempty set of *possible worlds* W, an *evaluation world* w, an *accessibility relation* R, and an *interpretation* $\mathcal{I}$ assigning each sentence letter to a truth-value at each world. Adding these resources improved on Carnap's [10, 11] semantic theories that evaluated sentences at *state-descriptions* which, for any atomic sentence of the language, include that sentence or its negation but not both. Carnap [10] specified how state-descriptions are to be understood:

> A state description is a class of sentences which represents a possible specific state of affairs by giving a complete description of the universe of individuals with respect to all properties and relations designated by predicates in the system.[8] (p. 50)

Since Carnap's state-descriptions are entirely syntactic in their construction and so determined by the primitive symbols included in a non-modal language, Carnap does not provide a genuine model theory but rather a single structure by which to interpret a modal language. Building on Carnap's efforts, Kripke [13] first sought to evaluate sentences at *complete assignments* over a *domain* D where each assignment maps the

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[6] I present a hyperintensional task semantics in [6], defining world states as maximal possible states.

[7] The Lean 4 repository https://github.com/benbrastmckie/BimodalLogic for this paper implements the soundness and completeness results as well as a small library of derivations.

[8] In his earlier work, Carnap [12, p. 95] writes, "If a false sentence is not L-false, hence not self-contradictory, it describes a situation which [is] possible though not real," and later that, "a system S has to do with many objects, and hence we have to consider the possible states of affairs of all the objects dealt with in S and with respect to all properties, relations, etc., dealt with in S," (p. 101) indicating Carnap's concern with an objective modality that quantifies over *possible circumstances* rather than the interpretational modality that the formal details of Carnap's theory are better equipped to support.

singular terms to elements in D and n-adic predicates to sets of n-tuples of elements in D where sentence letters are assigned to truth or falsity. By contrast with Carnap's state-descriptions which belong to a single fully specified structure, Kripke took the domain D included in a model to be any set whatsoever, where it was by quantifying over all such models that Kripke defined validity for the language. Given that there are no two complete assignments that agree on all elements of the language, the range of complete assignments is determined by the primitive symbols included in the language together with the domain provided by the models of the language.

Despite the language relativity of both Carnap's state-descriptions and Kripke's complete assignments modeled after them, these constructions sought to represent *possibilities* to provide semantic clauses for modal operators by quantifying over all or some possibilities. Whereas objective modality concerns possible ways for a system to be configured independent of the expressive resources included in any language, *interpretational modality* concerns consistent ways of interpreting a language, making a degree of language-relativity both inherent and appropriate. Although the language relativity of Carnap [10, 11] and Kripke's [13] early accounts presents no issue for the interpretational modals that they may be taken to have described, Kripke sought to accommodate a broader range of readings for the modal operators by decoupling the possibilities over which the modal operators quantified from the primitive symbols in the language.[9] Beginning with the propositional fragment, Kripke [8] introduced a primitive set of *possible worlds* W which he took to be any nonempty set whatsoever, where the interpretation of the language was handled by an *interpretation function* mapping each sentence letter and possible world to a truth-value.[10] By also specifying a primitive relation R over W for *relative possibility*, Kripke showed how to associate the reflexivity, symmetry, and transitivity constraints on R with the corresponding **T**, **B**, and **4** axioms which characterized the differences between the most prominent modal systems that Lewis and Langford [7] had introduced.[11] Although Kripke [9] went on to extend his semantics to include predicates and first-order quantifiers, it will suffice for present purposes to restrict consideration to the propositional fragment of the languages with which Carnap and Kripke were concerned.

Despite their differences, neither Carnap nor Kripke were focused on interpreting languages with tense operators or singular terms for an ontology of times.[12] Insofar as state-descriptions, complete assignments, and possible worlds are taken to determine the truth-values of temporary sentences, it is implausible to interpret these elements as temporally extended histories. Besides being irrelevant to the interpretation of a modal language without tense operators, taking possible worlds to be temporally extended

[9]Kripke [14, 15] went on to describe a metaphysical reading of the modal operators for which it is inappropriate to relativize the range of possibilities to the expressive power of a language.

[10]Officially, Kripke [8] took the *models* on a *model structure* $\langle W, R, w\rangle$ to be functions which take a propositional variable and world to a truth-value. I have replaced propositional variables with sentence letters and reserved the label 'model' to improve consistency with the definitions presented below.

[11]The relational structures underlying Kripke semantics were anticipated algebraically by Jónsson and Tarski's representation theorem for Boolean algebras with operators [16, 17], which embeds every such algebra into a powerset algebra over a set with a binary relation— i.e., a frame. However, neither Tarski nor his contemporaries recognized the connection to modal semantics. Tarski reportedly told Kripke in 1962 that he could not see a link between their respective programs. See Goldblatt [18] for historical discussion.

[12]Although Kripke interprets possible worlds as *moments* in a letter discussing Prior's [19] temporal interpretation of the modal operators, he writes, "I myself was working with ordinary modal logic." For the published correspondence and accompanying discussion, see Ploug [20, p. 373].

underdetermines the truth-values for temporary sentences, requiring the addition of a temporal parameter to the point of evaluation in order to specify the time in the possible world at which the sentence is to be evaluated. However, neither Carnap nor Kripke provide any indication that times are to be considered in evaluating sentences. Since the possible worlds that Kripke took to be primitive cannot be interpreted as temporally extended, they must be instantaneous.[13]

Although it is natural to conceive of possible worlds as complete configurations of a system at a moment when interpreting the sentences of a language with either tense operators or modal operators but not both, the same cannot be said for bimodal languages with both tense and modal operators. Not only do instantaneous moments fail to specify what comes before or after when considered on their own, including an ordering of moments once and for all fails to capture the range of different orderings of moments that are possible. What is needed to interpret bimodal sentences is an encoding of both the modal and temporal dimensions of the semantics.

While working to develop a semantics for tense logic, Prior [19, 21] took the first steps towards constructing the resources needed to interpret a bimodal language. Rather than restricting consideration to operators that quantify over instantaneous configurations of a system, Prior defined world histories to consist of temporally ordered configuration states. In addition to the tense operators which quantify over the past and future states in a history, Prior introduced operators that quantify over world histories to characterize the open future. The following section will review Prior's efforts, highlighting his most important insight that world histories are to be defined rather than primitive while also considering the limitations this approach faces.

2.1 World States

In *Time and Modality*, Prior [19] developed a Diodorean interpretation of $\Diamond_{\mathrm{F}}\varphi$ at a sequence of numbers which he took to represent the future.[14] Whereas the first number in the sequence represented the truth-value of φ in the present, the subsequent numbers represent the truth-value of φ at incrementally later times.

Commenting on Prior's book, Kripke observed in a letter [20, p. 370] that the Diodorean system validates $\boxed{\mathrm{F}}\Diamond_{\mathrm{F}}\varphi \vee \boxed{\mathrm{F}}\Diamond_{\mathrm{F}}\neg\varphi$ which does not belong to an S4 logic. Kripke went on to suggest a branching structure in which the past is determined but the future remains open, where models of this kind are more appropriate to the S4 logic that Prior had discussed. Crediting Kripke for this account in [21, p. 27], Prior developed a number of semantic theories and corresponding logics for tensed languages where sentences are evaluated at *world states* which model, "instantaneous total states of the world," (p. 88) rather than temporally extended histories. Intuitively, each world state is a complete configuration of the system under study at an instant,

[13] In the only example that Kripke [9] provides, the sentence 'Sherlock Holmes is bald' has a truth-value even though a time has not been specified and the name 'Sherlock Holmes' does not refer. Assuming Cary to be in the flesh and blood, 'Cary is reading' may be said to have a truth-value with considerably less controversy. In both cases, Kripke's semantics returns a truth-value despite the fact that a time has not been included in the sentence in question nor among the parameters at which that sentence is interpreted.

[14] Diodorus Cronus (died c. 284 BCE) was a Megarian logician who defined possibility and necessity in temporal terms, where a proposition is possible just in case it either is or will be true, and necessary just in case it is true and always will be true. Although Prior was concerned to provide a semantics and logic for a tensed language without metaphysical modal operators, he stops short of claiming that the metaphysical modals are definable in terms of the temporal operators with which he was concerned.

individuated by the intrinsic properties of that system in question. In order to interpret the tense operators, Prior included a strict *earlier-than* relation $<$ to order the world states. Situating his semantic approach in contrast to previous developments, Prior [21, p. 7-8] cites Broad's [22, p. 315] criticism of McTaggart's [23] famous argument that time is unreal, writing that the fundamental error in McTaggart's argument is his attempt to, "impose conditions appropriate to a tenseless language upon a tensed one." Prior [21, p. 12] avoids this defect by following Broad's suggestion to, "drop the temporal predicates 'past', 'present', and 'future'."[15] Whereas McTaggart includes times, temporal predicates, and tense operators all in one language in order to present his arguments, Prior interprets an object language with tense operators by appealing to an extensional metalanguage in which world states are ordered by an earlier-than relation but no other temporal relations or properties are included.

Although the world states form a strict total order in one of the simplest versions of his semantics, Prior goes on to consider models which accommodate an open future by taking the world states to form a strict partial order that is connected and left-linear rather than total.[16] Despite these differences, $<$ is a strict partial order on each account that Prior considered since no world state is permitted to be earlier than itself, and a world state s is earlier than r if both s is earlier than t and t is earlier than r. It is important to observe that taking world states to be strictly ordered prevents the same world state from occurring more than once in any history for that system. However, supposing that history never repeats itself is a substantive assumption which should not be built into the semantics. Moreover, there are systems which we may wish to study that admit loops in their evolution. For instance, taking the system in question to be a chessboard, there are histories for that system which include meandering end games where the same board state occurs more than once. Nothing should rule out consideration of such games of chess. Rather, this example highlights a fundamental limitation of the theoretical roles that world states play in Prior's semantics. Although it is natural to consider systems which occupy the same instantaneous configuration at different times, this is forbidden if world states are strictly ordered.

To disentangle the distinct theoretical roles which world states play in Prior's semantics, it will help to define a minimally constrained class of models that generalize on Thomason's [25] reconstruction of the Peircean and Ockhamist semantic theories that Prior [21] developed. Letting $\mathcal{L}^{\mathrm{T}} \coloneqq \langle \mathbb{L}, \bot, \rightarrow, \boxed{\scriptstyle P}, \boxed{\scriptstyle F} \rangle$ be a non-modal propositional language, a *strict model* of $\mathcal{L}^{\mathrm{T}}$ is an ordered triple $\mathcal{P} = \langle T, <, |\cdot| \rangle$ where $\langle T, < \rangle$ is a strict partial order for a nonempty set of world states T, and $|p_i| \subseteq T$ for all sentence letters $p_i \in \mathbb{L}$.[17] Although some strict models correspond to a single history for a system by also being total, strict models may accommodate distinct histories which do not intersect or only partially overlap. A *strict history* in a strict model $\langle T, <, |\cdot| \rangle$

[15] Instead of following Reichenbach [24] in distinguishing between the point-of-speech, point-of-reference, and point-of-event to evaluate tensed sentences at a single world state, it is by evaluating sentences with nested tense operators that the evaluation time shifts between the right number of world states.

[16] A *strict partial order* $\langle T, < \rangle$ is a set T equipped with an irreflexive and transitive relation $<$. Letting $x \sim y \coloneqq (x < y) \vee (x = y) \vee (x > y)$ express that x and y are comparable, a strict partial order $\langle T, < \rangle$ is *total* if $s \sim t$ for any $s, t \in T$. A strict partial order $\langle T, < \rangle$ is *left-linear* just in case $s \sim t$ for any $s, t, r \in T$ where $s < r$ and $t < r$. Letting $\stackrel{*}{\sim}$ be the transitive closure of $\sim$, a frame $\langle T, < \rangle$ is *connected* just in case $x \stackrel{*}{\sim} y$ for all $x, y \in T$. Writing $z \leqslant x$ for $z < x \vee z = x$, a left-linear frame is connected just in case for any $x, y \in T$ there is some z where $z \leqslant x$ and $z \leqslant y$.

[17] I will assume $\neg\varphi \coloneqq \varphi \rightarrow \bot$, $\varphi \vee \psi \coloneqq \neg\varphi \rightarrow \psi$, $\varphi \wedge \psi \coloneqq \neg(\varphi \rightarrow \neg\psi)$, and $\varphi \leftrightarrow \psi \coloneqq (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

is any maximal total suborder $h_i = \langle T_i, <_i \rangle$ of $\langle T, < \rangle$.[18] Letting $H_{\mathcal{P}}$ be the set of all strict histories of a strict model $\mathcal{P}$, we may take $H^x_{\mathcal{P}} \coloneqq \{\langle T_i, <_i \rangle \in H_{\mathcal{P}} \mid x \in T_i\}$ to be the strict histories $\langle T_i, <_i \rangle$ of $\mathcal{P}$ which include the world state $x \in T_i$.

Despite being defined rather than primitive, strict histories play a similar role to possible worlds in the semantics that Montague [1] and Kaplan [2] went on to provide since each strict history corresponds to a complete temporal evolution of the system in question. In particular, consider the following semantic clauses:

Peircean: $\mathcal{P}, x \vDash \boxed{\scriptstyle\mathrm{P}}^{\mathrm{P}}\varphi$ *iff* $\mathcal{P}, y \vDash \varphi$ for some $h_i \in H^x_{\mathcal{P}}$ and all $y \in h_i$ where $y <_i x$.
$\mathcal{P}, x \vDash \boxed{\scriptstyle\mathrm{F}}^{\mathrm{P}}\varphi$ *iff* $\mathcal{P}, y \vDash \varphi$ for some $h_i \in H^x_{\mathcal{P}}$ and all $y \in h_i$ where $x <_i y$.

Ockhamist: $\mathcal{P}, h_i, x \vDash \boxed{\scriptstyle\mathrm{P}}^{\mathrm{O}}\varphi$ *iff* $\mathcal{P}, h_i, y \vDash \varphi$ for all $y \in T_i$ where $y <_i x$.
$\mathcal{P}, h_i, x \vDash \boxed{\scriptstyle\mathrm{F}}^{\mathrm{O}}\varphi$ *iff* $\mathcal{P}, h_i, y \vDash \varphi$ for all $y \in T_i$ where $x <_i y$.

Whereas Prior [21, p. 126, 132] provided a semantics for the metric tense operators and Thomason [25] restricted attention to $\langle\mathrm{P}\rangle^{\mathrm{P}}$ and $\langle\mathrm{F}\rangle^{\mathrm{P}}$, I have derived the semantics for $\boxed{\scriptstyle\mathrm{P}}^{\mathrm{P}}$ and $\boxed{\scriptstyle\mathrm{F}}^{\mathrm{P}}$ from the semantics that Thomason provides while generalizing the account to accommodate all strict models of $\mathcal{L}^{\mathrm{T}}$.[19] The Peircean and Ockhamist semantics disagree about what the tense operators express at any world state that belongs to more than one strict history. Whereas the Peircean semantics quantifies over all past or future world states in *some* strict history that includes the world state at which the tensed sentence is evaluated, the Ockhamist semantics evaluates tensed sentences at both a strict history and world state in that history, quantifying over just the world states in that strict history. As a result, the Peircean semantics has a number of unnatural consequences. For instance, given a board state in a game of chess where there is at least one history in which Black does not make any further blunders, the sentence 'Black is not going to blunder' is true even though there may be other histories in which Black goes on to blunder. This is far from natural. Moreover, revising the Peircean semantics to quantify over *all* strict histories in addition to all past or future world states makes the operators $\boxed{\scriptstyle\mathrm{P}}^{\mathrm{P}}$ and $\boxed{\scriptstyle\mathrm{F}}^{\mathrm{P}}$ too strong and their duals too weak. For instance, in a game of chess in which there is at least one future in which the white king is in checkmate and at least one future in which the black king is in checkmate, both 'Black is going to win' and 'White is going to win' come out true.

Rather than quantifying over all past or future world states in either some or all strict histories, the Ockhamist semantics evaluates sentences at both a strict history and world state. The sentences 'Black is going to win' and 'White is going to win' cannot both be true since there is no way for both kings to be in checkmate in the same game. In addition to claiming this advantage, another prominent difference between the Ockhamist and Peircean semantics is the expressive power that they each afford. Since the Ockhamist tense operators only quantify over the world states in the strict history included in the point of evaluation, the Peircean operators may be defined in terms of the Ockhamist operators given the following *stability operator*:

[18] Letting a *total suborder* of $\langle T, < \rangle$ be any total order $\langle T_i, <_i \rangle$ where $T_i \subseteq T$ and $<_i$ is the restriction of $<$ to T_i, a total suborder $\langle T_i, <_i \rangle$ of $\langle T, < \rangle$ is *maximal* just in case $\langle T_j, <_j \rangle = \langle T_i, <_i \rangle$ for any total suborder $\langle T_j, <_j \rangle$ of $\langle T, < \rangle$ where $\langle T_i, <_i \rangle$ is a total suborder of $\langle T_j, <_j \rangle$.

[19] The Peircean semantics given above can be simplified were one to restrict consideration to strict models which are left-linear (similarly right-linear), since world states would then have a unique past (future).

($\boxdot^{\mathrm{O}}$) $\mathcal{P}, h_i, x \vDash \boxdot^{\mathrm{O}}\varphi$ *iff* $\mathcal{P}, h_j, x \vDash \varphi$ for all $h_j \in H^x_{\mathcal{P}}$.

By contrast to the Ockhamist tense operators, the stability operator quantifies over all intersecting strict histories that occupy the same world state at which the sentence is evaluated. Prior [21, p. 125] reads '$\boxdot^{\mathrm{O}}\varphi$' as 'It is now unpreventable that φ', though uses 'Necessarily φ' for convenience, claiming that whereas the Peircean's, "rather strong 'will be' is simply the Ockhamist 'necessarily will be', the Ockhamist 'will be' [is] untranslatable," (p. 130) for the Peircean.[20] As Prior observes, $\varphi \to \boxdot^{\mathrm{O}}\varphi$ is valid for non-temporal sentences, where $\boxed{\scriptstyle\mathrm{P}}$ (similarly $\boxed{\scriptstyle\mathrm{F}}$) may be included in φ by restricting to strict models that are left-linear (right-linear). Although the Ockhamist may define the Peircean tense operators $\boxed{\scriptstyle\mathrm{P}}^{\mathrm{P}}\varphi \coloneqq \diamonddot^{\mathrm{O}}\boxed{\scriptstyle\mathrm{P}}^{\mathrm{O}}\varphi$ and $\boxed{\scriptstyle\mathrm{F}}^{\mathrm{P}}\varphi \coloneqq \diamonddot^{\mathrm{O}}\boxed{\scriptstyle\mathrm{F}}^{\mathrm{O}}\varphi$, the Peircean is not in a position to define the Ockhamist operators for tense, nor to provide a semantics for a stability operator since sentences are evaluated at world states alone.

Despite their differences, neither the Peircean nor Ockhamist semantics permit the same world state in T to occur more than once in a strict history, nor are there strict histories in which the same world states in T occur in different orders. Rather, $\langle T, < \rangle$ is required to be a strict partial order for any strict model $\langle T, <, |\cdot| \rangle$ of $\mathcal{L}^{\mathrm{T}}$, and so if $s <_i t$ for any strict history of $\langle T, <, |\cdot| \rangle$, then $t \nless_j s$ for all strict histories of $\langle T, <, |\cdot| \rangle$. However, given Prior's conception of world states as complete configurations, there are systems in which the same world states occur more than once or in different orders in different possible histories. As brought out above, a chess game may include the same board state more than once, or two chess games may agree in all respects with the exception of a transposition of board states which occur in a different order.[21]

Insofar as T is taken to be the set of all world states of a particular system under study, the strict histories for that system do not permit the same world states to occur more than once or in different orders, significantly compromising the range of applications for the semantics. Rather than accepting these limitations, it is natural to reject Prior's conception of T as the set of world states where these are taken to be complete configurations of the system, taking T to be the set of *times* in accordance with their strict ordering. After all, T is intended to provide resources for articulating a semantics for temporal operators which quantify over what comes before or after each element in T. On this reading, the truth-conditions for the sentences of $\mathcal{L}^{\mathrm{T}}$ specify *when* a sentence is true without indicating which world state the system happens to occupy at each time. Strict models may include strict histories in which the same world states occur more than once or in different orders for the simple reason that strict models only concern the times at which sentences are evaluated without saying anything about the world states that the system occupies at those times. As a result, there is no telling which times in T occupy the same world states, or which strict histories include the same world states in different orders. For instance, supposing the white king to be in check at multiple times in a game of chess, there is no way to distinguish which times in the truth-condition for 'The white king is in check' correspond to the same complete configuration of the chessboard (if any) and which times correspond to distinct configurations. Moreover, given two strict histories for a chess game which

[20] Prior [21, p. 130] attributes these observations to J.M. Shorter in 1957 but does not provide a proof.

[21] Following Müller's [26] suggestion, Rumberg [27] extends Prior [21] and Thomason's [25] semantics. By virtue of working over a partial order, this extension faces the same difficulties brought out here.

agree in all respects with the exception of a short sequence of moves in which the same board states occur in different orders, the divergent sequences must include distinct times in T despite reordering the same configurations of the chessboard.

Taking T to be the set of times rather than world states avoids ruling out strict histories in which the same world states occur more than once or in different orders. Since all that is required to interpret tense operators is to specify *when* each sentence is true, the erroneous interpretation of T as a set of complete configurations of a system may be forgiven when considering non-modal languages like $\mathcal{L}^{\mathrm{T}}$. Despite providing an adequate range of semantic primitives for interpreting the tensed languages with which Prior and Thomason were concerned, the same cannot be said for a bimodal language with operators for both tense and metaphysical modality. In order to evaluate a sentence φ of the bimodal language $\mathcal{L}$ at a model $\mathcal{P}$, strict history h_i, and time x, it is important that $x \in T_i$ be a valid time in the strict history $h_i = \langle T_i, <_i \rangle$. As a result, the strongest modal operator that an Ockhamist model theory can provide only quantifies over those strict histories in which the time of evaluation occurs, and so cannot quantify over the full range of strict histories included in a strict model. This is exactly what the Ockhamist stability operator $\boxdot^{\mathrm{O}}$ achieves, though Prior is careful not to confuse this modality with metaphysical necessity since the stability modal does not quantify over all strict histories whatsoever. Since there is no Ockhamist modality that is able to quantify over all strict histories, the range of strict histories cannot represent the full range of possible evolutions for the system in question.

The source of these limitations is that Prior takes the elements in T to answer two different questions: *when* a sentence is true which is given by a position in a strict order, and *what* configuration the system then occupies. Since the position-marker and the configuration are one and the same, no configuration can occupy two positions, which is why the same world state cannot recur in any strict history. Rather than accepting these limitations, I will take both *world states* and *durations* to be semantic primitives in §3, separating the theoretical roles that Prior conflates. By also including a *task relation*, I will define possible worlds as appropriately constrained functions from durations to world states, tracing out different paths through the space of all world states in alignment with standard definitions in dynamical systems theory. In addition to providing a natural model theory in which possible worlds may occupy the same world states more than once or in different orders, taking sets of world states to provide truth-conditions for the sentence letters in the language makes times exogenous to the interpretation of the language. As I show in **C6** of the *Appendix*, both **P1** and **P2** are valid over a natural class of models that I define for the bimodal language.

Instead of constructing the space of possible worlds from world states, tasks, and times, Montague [1] and Kaplan [2] separate the *when* from the *what* by taking both the set of possible worlds W and the set of times T to be primitive along with a weak total order $\leqslant$ for the *at least as early as* relation.[22] Abstracting from their differences, I will take a *two-dimensional model* $\mathcal{M}_2 = \langle W, T, \leqslant, |\cdot| \rangle$ to include an interpretation function where $|p_i| \subseteq W \times T$ is a set of world-time pairs for each sentence letter p_i with $i \in \mathbb{N}$. However natural it may seem to extend Kripke's semantics

[22] Little turns on whether the order is weak $\leqslant$ or strict $<$ following Prior. I will follow Montague and Kaplan in omitting consideration of the accessibility relation between possible worlds.

along these lines, Montague's semantics undermines the expressive power of the modal operator included in his language while Kaplan's semantics invalidates the perpetuity principles, weakening the logic that the resulting model theory supports. The following subsections will review the shortcomings of these early bimodal semantic theories before turning to present the construction of possible worlds in **§3**.

2.2 Necessarily Always

Whereas Montague [1] sought to provide a formal semantics for a fragment of English by regimenting that fragment in a quantified bimodal language for which he gave a recursive semantics, I will restrict consideration to the propositional fragment of Montague's formal language. To facilitate comparison, I will include $\boxtimes$ in place of $\Box$ in the propositional language $\mathcal{L}^{\text{M}} = \langle \mathbb{L}, \bot, \rightarrow, \boxtimes, \boxed{\text{P}}, \boxed{\text{F}} \rangle$. By defining the well-formed sentences of $\mathcal{L}^{\text{M}}$ in the usual way, we may extend the interpretation provided by each two-dimensional model $\mathcal{M}_2$ to all well-formed sentences of $\mathcal{L}^{\text{M}}$ as follows:

(p_i) $\mathcal{M}_2, w, x \vDash p_i$ *iff* $\langle w, x \rangle \in |p_i|$.

($\bot$) $\mathcal{M}_2, w, x \nvDash \bot$.

($\rightarrow$) $\mathcal{M}_2, w, x \vDash \varphi \rightarrow \psi$ *iff* $\mathcal{M}_2, w, x \nvDash \varphi$ or $\mathcal{M}_2, w, x \vDash \psi$.

($\boxtimes$) $\mathcal{M}_2, w, x \vDash \boxtimes\varphi$ *iff* $\mathcal{M}_2, u, y \vDash \varphi$ for all $u \in W$ and $y \in T$.

($\boxed{\text{P}}$) $\mathcal{M}_2, w, x \vDash \boxed{\text{P}}\varphi$ *iff* $\mathcal{M}_2, w, y \vDash \varphi$ for all $y \in T$ where $y < x$.

($\boxed{\text{F}}$) $\mathcal{M}_2, w, x \vDash \boxed{\text{F}}\varphi$ *iff* $\mathcal{M}_2, w, y \vDash \varphi$ for all $y \in T$ where $x < y$.

Postponing the controversies that beset the interpretation of the future tense operator to **§4.1**, I will take the semantics for both the extensional and tense operators to be uncontroversial for the time being. Whereas Montague [1] took modal claims of the form $\ulcorner\boxtimes\varphi\urcorner$ to be read ⌜It is necessarily the case that φ⌝, the same paper in the edited collection included an editor's note by Richard Thomason [28, p. 259, FN 9] that $\boxtimes$, "is interpreted in the sense of 'necessarily always'." After all, $\boxtimes\varphi$ is true in a world w at a time x just in case φ is true in all worlds at all times, quantifying over both the modal and temporal dimensions of Montague's semantics at once.

By letting a *moment* be any ordered pair $\langle w, x \rangle \in W \times T$, the semantics for $\boxtimes$ quantifies over all moments.[23] By contrast, I will take the following *universal semantics* for $\boxed{2}$ to quantify over all worlds while leaving the temporal parameter unchanged:

($\boxed{2}$) $\mathcal{M}, w, x \vDash \boxed{2}\varphi$ *iff* $\mathcal{M}, u, x \vDash \varphi$ for all $u \in W$.

To evaluate whether $\boxed{2}$ or $\boxtimes$ expresses metaphysical necessity, I will take $\mathcal{L}^{\text{K}}$ to be the result of replacing $\boxtimes$ in $\mathcal{L}^{\text{M}}$ with $\boxed{2}$ while maintaining the other operators included in the language. Next we may derive the semantic clause for $\triangle$ from its definition above:

($\triangle$) $\mathcal{M}, w, x \vDash \triangle\varphi$ *iff* $\mathcal{M}, w, y \vDash \varphi$ for all $y \in T$.

[23] A moment here is a pair whose first coordinate is a primitive world. By constructing possible worlds in §3, the world coordinate is a function from durations to world states so that a moment fixes the world state that the possible world occupies at that duration from the origin in that possible world.

Extending $\mathcal{L}^{\text{K}}$ to include $\boxtimes$ as a primitive symbol makes $\boxtimes\varphi \leftrightarrow \boxed{2}\triangle\varphi$ valid by being true in every world at every time on any two-dimensional model, thereby rendering $\boxtimes$ redundant. Rather, I will maintain $\boxtimes\varphi \coloneqq \boxed{2}\triangle\varphi$ as a metalinguistic abbreviation in $\mathcal{L}^{\text{K}}$, deriving the semantics for $\boxtimes$ from its definition. Since **T5** in the *Appendix* proves that $\boxed{2}$ cannot be defined in $\mathcal{L}^{\text{M}}$, it follows that $\mathcal{L}^{\text{K}}$ is more expressive than $\mathcal{L}^{\text{M}}$. For the purposes of comparing $\boxed{2}$ and $\boxtimes$ in a common language, $\mathcal{L}^{\text{K}}$ enjoys a clear advantage over working in $\mathcal{L}^{\text{M}}$. Assuming that $\mathcal{L}^{\text{M}}$ is to be replaced by $\mathcal{L}^{\text{K}}$ where $\boxtimes$ is then defined in terms of $\boxed{2}$ and $\triangle$ as above, the question remains whether or not to read $\ulcorner\boxed{2}\varphi\urcorner$ as $\ulcorner$It is necessarily the case that $\varphi\urcorner$ in accordance with Thomason's suggested reading of $\ulcorner\boxtimes\varphi\urcorner$ as $\ulcorner$It is necessarily always the case that $\varphi\urcorner$, or to follow Montague.[24]

In opposition to Thomason's reading, one might appeal to the invalidity of **P1** and **P2** over the class of two-dimensional models in defense of taking $\boxtimes$ and $\Diamond_{\times}\varphi \coloneqq \neg\boxtimes\neg\varphi$ to be the metaphysical modals rather than $\boxed{2}$ and $\Diamond_{2}\varphi \coloneqq \neg\boxed{2}\neg\varphi$.[25] So long as there are multiple times in T, we may consider a model $\mathcal{M}_2$ where p_i is assigned to a set of world-time pairs that includes $\langle w', x\rangle$ for all $w' \in W$ at a fixed time $x \in T$ but that does not include $\langle w, x'\rangle$ for all times $x' \in T$ at a fixed world $w \in W$. It follows that $\boxed{2}p_i$ may be true in w at x while $\triangle p_i$ is false, thereby invalidating **P1** where **P2** is equivalent. By contrast, the following *trivial perpetuity principles* which result from replacing $\boxed{2}$ and $\Diamond_{2}$ in **P1** and **P2** with $\boxtimes$ and $\Diamond_{\times}$ may be shown to be valid:

TP $\boxtimes\varphi \rightarrow \triangle\varphi$. **CT** $\triangledown\varphi \rightarrow \Diamond_{\times}\varphi$.

Given the definition $\boxtimes\varphi \coloneqq \boxed{2}\triangle\varphi$, **TP** is an instance of the **T** axiom $\boxed{2}\psi \rightarrow \psi$ where **CT** is equivalent. Instead of expressing substantive interaction principles for tense and modality, the trivial perpetuity principles are valid over the two-dimensional models of $\mathcal{L}^{\text{K}}$ for entirely modal reasons. To avoid appealing to the status of the operators in one language or another, we may put the point purely semantically as follows:

Triviality: $(\forall w \in W \forall x \in T : \mathcal{M}, w, x \vDash \varphi) \rightarrow (u \in W \rightarrow \forall x \in T : \mathcal{M}, u, x \vDash \varphi)$.
$(u \in W \wedge \exists x \in T : \mathcal{M}, u, x \vDash \varphi) \rightarrow (\exists w \in W \exists x \in T : \mathcal{M}, w, x \vDash \varphi)$.

The principles above are instances of the first-order theorems $\forall w\Psi \rightarrow \Psi[u/w]$ and $\Psi[u/w] \rightarrow \exists w\Psi$.[26] Despite including quantification over times, nothing in Ψ accounts for why the *Triviality* principles are valid. Rather, these principles are valid on account of universal instantiation and existential generalization with respect to quantification over the worlds in W independent of the quantification over times in Ψ. That **TP** and **CT** are valid over all two-dimensional models is no compensation for failing to articulate substantive interaction principles. By contrast, the universal semantics for $\boxed{2}$ and $\Diamond_{2}$ enjoys a clear advantage over the Montagovian semantics for $\boxtimes$ and $\Diamond_{\times}$ by drawing on their greater expressive power to make **P1** and **P2** substantive.

[24] Admittedly, it is not clear that Montague [1] was concerned with metaphysical modality, though the semantics he provides for $\boxtimes$ is completely unrestricted in quantifying over times and worlds.

[25] Dorr and Goodman [29, p. 636] present considerations along these lines, though they, "think it best to avoid 'world'-talk altogether in theorizing about temporary matters" (p. 646). In opposition to this perspective, I will assume that the semantic primitives included in the models of a language ought to provide the intuitive bedrock by which to define meaningful truth-conditions for the sentences of the language.

[26] Replace Ψ with either: (1) $w \in W \rightarrow \forall x \in T : \mathcal{M}, w, x \vDash \varphi$; or (2) $w \in W \wedge \exists x \in T : \mathcal{M}, w, x \vDash \varphi$.

Dorr and Goodman [29, pp. 635, 655] define immediate necessity in terms of their primitive necessity operator with the resources that Fine [30] provides. By including a countable set of *time variables* $\mathcal{V} \coloneqq \{t_i : i \in \mathbb{N}\}$ in the language which may be bound by first-order quantifiers and taking an *assignment* to be any function $g : \mathcal{V} \to T$ from time variables in $\mathcal{V}$ to times in T, Dorr and Goodman define $\Box$ as follows:

($\exists t$) $\mathcal{M}, w, x, g \vDash \exists t\varphi$ *iff* $\mathcal{M}, w, x, g' \vDash \varphi$ for some g' differing from g at most in t.

($\texttt{Pr}$) $\mathcal{M}, w, x, g \vDash \texttt{Present}(t)$ *iff* $g(t) = x$.[27]

Converse Definition: $\boxed{2}\varphi \coloneqq \exists t[\texttt{Present}(t) \wedge \boxtimes(\texttt{Present}(t) \to \varphi)]$.[28]

By taking $\mathcal{L}^{\text{F}}$ to extend $\mathcal{L}^{\text{M}}$ to include the Finean resources given above, **L4** in the *Appendix* shows that $\boxed{2}\varphi$ is true in world w at time x on assignment g just in case φ is true in world u at time x on assignment g for all worlds $u \in W$. Given this alignment with the universal semantic clause provided above for $\boxed{2}$, **T6** shows that the perpetuity principles are invalid over the two-dimensional models for $\mathcal{L}^{\text{F}}$.

Whereas $\boxtimes$ is easy to define in terms of $\boxed{2}$ and $\triangle$ in $\mathcal{L}^{\text{K}}$, the ideological complexity of the *Converse Definition* reflects the relative obscurity of $\boxed{2}$ from the perspective of the primitive expressions included in $\mathcal{L}^{\text{F}}$. Given the naturalness of the semantics for $\boxed{2}$, the complexity of the *Converse Definition* indicates how awkward it is to make do with the operator $\boxtimes$ whose semantic clause quantifies over moments in $W \times T$ instead of worlds in W. Although the extra ideology adds expressive power to $\mathcal{L}^{\text{F}}$, including temporal predicates and quantifiers over times abandons Prior's insight that much of the trouble that McTaggart [23] faced was due to cobbling together tense operators, temporal predicates, and first-order quantification over times all into one language. Following Prior, I will exclude temporal predicates and first-order quantification over times from the object language, limiting use of these resources to the metalanguage which I will use to articulate truth-conditions for the sentences of a bimodal language. In addition to avoiding the threat of making the temporal operators superfluous, this approach does not carry the same commitment to an ontology of times.

Instead of validating the perpetuity principles trivially by taking the semantics for the metaphysical modals to quantify over all moments, the following subsection considers attempts to validate the perpetuity principles by constraining the class of two-dimensional models. As I will argue, the resulting theory is committed to an absolute theory of time that cannot be avoided without undermining the significance of the truth-conditions for the sentences of the language. This will set the stage for **§3** which provides the task semantics, proving that **P1** and **P2** are valid without imposing any *ad hoc* constraints on the space of models for the language.

2.3 Absolute Time

Instead of taking the semantics for the metaphysical modals to quantify over moments in $W \times T$, Kaplan's [2] semantics for $\boxed{2}$ quantifies over just the possible worlds W in a

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[27] Dorr and Goodman [29, p. 637] admit that Fine's [30] use of propositional quantifiers to eliminate time variables from the expanded language behaves pathologically when $\boxed{2}$ is replaced by $\boxtimes$.

[28] Dorr and Goodman [29, p. 656] refer to $\boxed{2}$ here as *immediate necessity* and are clear that they nowhere endorse model-theoretic characterizations of metaphysical necessity, Montagovian or otherwise.

two-dimensional model of $\mathcal{L}^{\text{K}}$. Although Kaplan does not articulate a logic for his two-dimensional semantics, **P1** and **P2** are invalid over the two-dimensional models that Kaplan defines. Since **T7** shows that no frame constraint on $\langle W, T, \leqslant\rangle$ can validate the perpetuity principles without collapsing the temporal order to include at most one time, it remains to validate the perpetuity principles over the two-dimensional semantics by imposing a model constraint. In particular, **T8** shows that **P1** and **P2** are valid over the abundant two-dimensional models of $\mathcal{L}^{\text{K}}$ which I will define as follows:

> *Time-Shift:* The worlds $w, w' \in W$ are *time-shifted from x to y*— i.e., $w \approx_x^y w'$— in a model $\mathcal{M}_2$ of $\mathcal{L}^{\text{K}}$ *iff* there is an order automorphism $\bar{a} : T \to T$ where $y = \bar{a}(x)$ and the following holds for all sentence letters $p_i \in \mathbb{L}$ and times $z \in T$:
>
> $$\langle w', \bar{a}(z)\rangle \in |p_i| \;\Leftrightarrow\; \langle w, z\rangle \in |p_i|.$$ [29]
>
> *Abundance:* A two-dimensional model $\mathcal{M}_2$ of $\mathcal{L}^{\text{K}}$ is *abundant iff* for every $w \in W$ and $x, y \in T$, there is some $w' \in W$ that is time-shifted from x to y, i.e., $w \approx_x^y w'$.

Abundant models include all time-shifted worlds.[30] Paradigm examples of abundant models identify T with either the set of integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, or real numbers $\mathbb{R}$. As **T9** shows, time is unbounded in abundant models with at least two distinct times.[31] Nevertheless, it is natural to take some systems such as a game of chess to have a bounded set of times. That abundant models cannot accommodate bounded temporal orders is a significant limitation which the task semantics avoids.

Restricting to the abundant models validates the perpetuity principles without providing a corresponding frame constraint. By **T7** this is unavoidable: no condition on $\langle W, T, \leqslant\rangle$ validates **P1** and **P2** short of collapsing the temporal order. In giving up the correspondence between the perpetuity axioms and a condition on the frames, we give up the central explanatory virtue of relational semantics— that a logic's theorems are underwritten by the structural properties defining a corresponding class of frames.[32] In addition to this formal deficit, abundant models are required to include a vast set of primitive worlds. We may state this precisely with the following definition:

> *Temporary:* A sentence φ of $\mathcal{L}^{\text{K}}$ is *temporary* in a two-dimensional model $\mathcal{M}_2$ *iff* $\mathcal{M}_2, w, x \vDash \varphi$ and $\mathcal{M}_2, w, y \nvDash \varphi$ for some world w and times x and y.

Abundant models with temporary sentences include all merely temporal differences between worlds. For instance, given an abundant model $\mathcal{M}_2$ in which φ is temporary, there is a world w and times $x, y \in T$ where $\mathcal{M}_2, w, x \vDash \varphi$ and $\mathcal{M}_2, w, y \nvDash \varphi$, and so

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[29] An *order automorphism* on the structure $\langle T, \leqslant\rangle$ is a monotonic bijection $\bar{a} : T \to T$ where $\bar{a}(x) \leqslant \bar{a}(y)$ whenever $x \leqslant y$, shifting all times in T without changing their order.

[30] Not all models are abundant: letting $W = \{w\}$ and $T = \{0, 1\}$ where $|p_1| = \{\langle w, 0\rangle\}$, there is no automorphism on T mapping 0 to 1 that preserves truth-values, and so the model is not abundant.

[31] A weak partial order $\langle T, \leqslant\rangle$ is *bounded* if both of the following hold and *unbounded* if neither holds:

Bounded Below: There is some $y \in T$ where $y \leqslant x$ for all $x \in T$.

Bounded Above: There is some $y \in T$ where $x \leqslant y$ for all $x \in T$.

[32] The frame-incompleteness of *Abundance* is an instance of the general problem that interaction axioms in products of modal logics pose for frame definability (see Kurucz [31, p. 221]). By contrast, compare the correspondence theorems for the tense fragment given in **§3.3** following van Benthem [32].

by *Abundance* there is a world w' that is time-shifted $w \approx_x^y w'$ from x to y. As shown in **L6**, what is true in w at x is exactly the same as what is true in w' at y. Since $w \approx_x^y w'$, it follows more generally that there is some order automorphism $\bar{a} : T \to T$ where $w \approx_z^{\bar{a}(z)} w'$ for all $z \in T$ by **L5**. Again by **L6**, it follows that for every time $z \in T$ exactly the same sentences that are true in w at z are true in w' at $\bar{a}(z)$. Insofar as there are intended models where W represents a meaningful range of possible histories of events without ever representing the same history more than once, it follows that abundant models with temporary sentences include all merely temporal differences between otherwise indistinguishable histories of events.

I will take *temporal absolutism* to be the claim that there are merely temporal differences between otherwise indistinguishable histories— $w \approx_x^y w'$ for some $w \neq w'$— a thesis that I will assume it is natural to resist. In addition to the formal deficit indicated above, requiring the models of $\mathcal{L}^{\text{K}}$ to be abundant gives rise to an unfortunate trade-off: either one must embrace temporal absolutism or else admit an excess of possible worlds that represent the same possible history many times over. Although it is preferable to avoid redundancies among the semantic primitives included in a model, one might attempt to defend a restriction to the abundant models by taking the time-shifted worlds to be empty artifacts. However, insofar as sets of world-time pairs are to provide meaningful truth-conditions for the well-formed sentences of $\mathcal{L}^{\text{K}}$, the worlds in W cannot be entirely void of significance. For instance, an *abundance theorist* might attempt to maintain intelligible truth-conditions by abstracting from the merely temporal differences by way of the following definitions:

Time-Shifted Worlds: $w \approx w'$ *iff* $w \approx_x^y w'$ for some $x, y \in T$.

World Abstraction: $[w] \coloneqq \{w' \in W \mid w \approx w'\}$.

Possible Worlds: $W_\approx \coloneqq \{[w] \subseteq W \mid w \in W\}$.

By letting $[w]$ be the set of time-shifted worlds that represent the same possible world as w, an abundance theorist might claim that $W_\approx$ represents the range of genuinely distinct possible worlds rather than the primitive set W. Although an abundant model which admits temporary sentences is guaranteed to include merely temporal differences between the primitive worlds in W, the elements in $W_\approx$ abstract from the merely temporal differences between worlds. An abundance theorist may take **T8** to show how to validate **P1** and **P2** by restricting consideration to the abundant models without embracing absolutism or undermining the interpretation of possible worlds.

Despite providing a method for identifying a reduced range of genuinely distinct possible worlds for each abundant model of $\mathcal{L}^{\text{K}}$, it is important to observe that $W_\approx$ is defined in terms of the interpretation of the sentence letters of the language. Even if the worlds $w \approx w'$ are time-shifted in a model $\mathcal{M}_2$ of $\mathcal{L}^{\text{K}}$ and so $[w] = [w']$, it does not follow that $w \approx w'$ in any other model $\mathcal{M}_2'$ of $\mathcal{L}^{\text{K}}$. For instance, assuming $w \approx_x^y w'$ in $\mathcal{M}_2$ for just $x, y \in T$ where $x \neq y$ and $\langle w, x \rangle \in |p_0|$, there is an automorphism $\bar{a} : T \to T$ where $y = \bar{a}(x)$ and $\langle w', y \rangle \in |p_0|$. By letting $\mathcal{M}_2'$ be identical to $\mathcal{M}_2$ except for taking $\langle w', y \rangle \notin |p_0|$, it follows that $w \not\approx_x^y w'$ in $\mathcal{M}_2'$ for any $x, y \in T$, and so $[w] \neq [w']$ in $\mathcal{M}_2'$. This construction demonstrates how the equivalence classes in $W_\approx$ may expand or contract depending on the truth-conditions assigned to the sentence letters by a model $\mathcal{M}_2$ of the language $\mathcal{L}^{\text{K}}$. The same construction shows that *Abundance* itself may be destroyed by changing the interpretation of a single sentence letter, confirming that it is a condition on models rather than frames since it is not preserved under a change of valuation. As a result, an abundance theorist cannot appeal to $W_{\approx}$ in order to specify meaningful truth-conditions for the sentence letters of $\mathcal{L}^{\text{K}}$ without circularity. In particular, consider the following definitions:

Time-Shifted Moments: $[w, x] \coloneqq \{\langle u, y\rangle \mid w \approx^y_x u\}$.

Truth-Conditional Abstraction: $|p_i|_{\approx} \coloneqq \{[w, x] \mid \langle w, x\rangle \in |p_i|\}$.

The first definition identifies the genuinely distinct moments in $\mathcal{M}_2$ by abstracting from the differences between moments which make the same sentence letters true in $\mathcal{M}_2$. However, for $|p_i|_{\approx}$ to provide a meaningful truth-condition for p_i in $\mathcal{M}_2$, one must already have an independent grasp of the original truth-condition $|p_i|$ provided by $\mathcal{M}_2$, undermining the need to specify $|p_i|_{\approx}$ in the first place. In addition to circularity, taking the truth-conditions for the sentence letters of $\mathcal{L}^{\text{K}}$ to be the basis upon which to identify the range of genuinely distinct possible worlds $W_{\approx}$ puts the cart before the horse. Rather, the *intended models* of the language aim to provide a range of meaningful semantic primitives by which to specify truth-conditions for the language, not the other way around. Whereas an *instrumentalist* gives up the ambition to admit intended models, I will take it to be an advantage of any semantic theory to avoid undermining the interpretation of the semantic primitives for that theory.

It is worth comparing a similar strategy applied to the semantics of a first-order extensional language $\mathcal{L}^1$. Given a domain D that represents a meaningful range of primitive *objects*, a truth-conditional semantics may interpret $\mathcal{L}^1$ by specifying the extensions of the constants and n-place predicates as elements of D and subsets of D^n. For instance, we may interpret a constant c in a model by assigning it to an element of D and interpret a one-place predicate F in a model by taking its extension to be a subset of the domain D. However, if it is denied that D includes genuinely distinct objects but rather some other entities which may represent the same genuinely distinct object many times over, we lose our initial grasp on the meaning of the constant c and the predicate F. If D is left uninterpreted, or else interpreted in terms of another assignment of constants and predicates to extensions in D^n, then we cannot look to D to provide an independent basis upon which to interpret the language $\mathcal{L}^1$.

In keeping with a standard methodology in truth-conditional semantics, I will take the semantic primitives in an intended model for a given object language to provide an independently meaningful basis upon which to make sense of the truth-conditions for that language. Rather than presuming that the only way for the semantic primitives in an intended model to be meaningful is to be identical to the parts of the reality that they model, I will take the intended models of a language to provide an idealization that simulates what that object language is able to express with the resources of a well-theorized metalanguage. Whereas the object language is of primary metaphysical significance, the metalanguage provides expressive resources for simulating what the object language is able to express rather than a description of the entities to which the object language is implicitly committed. I will refer to this approach as *simulation* *metasemantics* in opposition to *realist metasemantics* which takes the intended model to include the constituents of reality that the object language describes as well as to *instrumentalist metasemantics* in which the semantic primitives have no significance beyond the instrumental role that they play in the semantics.[33]

To validate the perpetuity principles without undermining the significance of the truth-conditions for the language, the following section presents an alternative to two-dimensional semantics that is compatible with a simulation metasemantics. Instead of taking possible worlds to be primitive points devoid of any internal structure, I will provide a *task semantics* that constructs possible worlds from *world states*, *tasks*, and *durations*. As I show, the resulting models provide a natural first-order simulation of what the bimodal language is able to express while also validating **P1** and **P2**.

3 Possible Worlds

To interpret the bimodal language $\mathcal{L} \coloneqq \langle \mathbb{L}, \bot, \rightarrow, \Box, \boxed{\scriptstyle P}, \boxed{\scriptstyle F} \rangle$, well-formed sentences will be evaluated at a *possible world* and a *duration* in addition to a model of $\mathcal{L}$. Rather than following Montague [1] and Kaplan [2] in taking possible worlds to be primitive, I will define possible worlds to be functions from durations to world states.[34] Whereas world states are the instantaneous maximal possible configurations of the system under study, the durations parameterize the possible trajectories through the space of world states. Accordingly, I will take the durations and not the world states to form a total order while positing additive group structure so that durations can be added and subtracted. Each duration x is used to indicate the world state a possible world occupies after x duration from the origin, where this justifies the convention of referring to durations as *times* relative to a possible world, often left implicit. The *interpretation function* included in a model of $\mathcal{L}$ maps each sentence letter in $\mathcal{L}$ to a set of world states, providing the truth-condition for that sentence by specifying the configurations of the system in which that sentence is true. Since each set of world states represents the ways for the system to be without including any temporal elements, durations are strictly exogenous to the interpretation of the sentence letters in language $\mathcal{L}$, where this fact plays a critical role in validating the perpetuity principles.

Positing a set of *world states* W in addition to a set of *durations* D distinguishes the states a system is in from when the system is in those states. A *temporal order* is defined over D to be any nontrivial totally ordered abelian group $\mathcal{D} = \langle D, +, 0, \leqslant \rangle$.[35] Instead of admitting the incomparable times permitted by a partial order, $\mathcal{D}$ is used

[33] Cresswell [33] shows that tense operators supplemented with selective indexical devices— operators that store and retrieve evaluation indices— are expressively equivalent to first-order quantification over times, extending the result to modal operators and possible worlds. Cresswell concludes that "the [ontological] commitment is genuine if the range of sentences in which these constructions occur requires a semantics equivalent in power to explicit variable binding" (p. 2). The bimodal language $\mathcal{L}^{\mathrm{K}}$ includes no such selective devices, and its operators are all unselective in Cresswell's sense. However, even if $\mathcal{L}^{\mathrm{K}}$ were enriched with such devices, I do not follow Cresswell in taking semantics to be any guide to ontological commitment.

[34] Although the world states will be primitive for our purposes, I define world states in Brast-McKie [6] by appealing to the task and parthood relations to provide a semantics for counterfactual conditionals.

[35] A *group* is any $\langle G, \cdot, 1_G \rangle$ where: (1) $a \cdot b \in G$ whenever $a, b \in G$; (2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$; (3) $1_G \in G$ where $a \cdot 1_G = 1_G \cdot a = a$ for all $a \in G$; and (4) for each $a \in G$, there is some $-a \in G$ where $a \cdot (-a) = 1_G$, written $a - a = 1_G$ for ease. A group is *nontrivial* just in case $a \neq 1_G$ for some $a \in G$. A group is *abelian* just in case $a \cdot b = b \cdot a$ for all $a, b \in G$. A group is *totally ordered* by $\leqslant$ just in case $\leqslant$ is a total order where for all $a, b, c \in G$, if $a \leqslant b$, then $a \cdot c \leqslant b \cdot c$.

to define the space of possible evolutions for a system where time is totally ordered in each. Since not every path through the space of world states is possible, frames include a *parameterized task relation* $w \Rightarrow_x u$ to encode which transitions between world states $w, u \in W$ over a positive duration $x \geqslant 0$ are possible. Defining reverse transitions by the *converse convention* $w \Rightarrow_{-x} u \coloneqq u \Rightarrow_x w$ for $x \geqslant 0$, the task relation determines the following for any world states $w, v \in W$ and durations $x, y \in D$:

Fiber: $\texttt{fib}(w, x) \coloneqq \{u \in W : w \Rightarrow_x u\}$.

Cone: $(w)_x \coloneqq \bigcup_{|y|<x} \texttt{fib}(w, y)$ where $x > 0$.

Segment: $[w, v]^y_x \coloneqq \texttt{fib}(w, x) \cap \texttt{fib}(v, -y)$ where $x, y \geqslant 0$.

Each fiber collects the world states accessible from some w over a duration x, where negative durations flip the direction of accessibility. Whereas a cone collects the world states accessible from w within x in either temporal direction by taking the union of all fibers for w over $(-x, x)$, a segment collects all world states u accessible from both w of x duration in the past and v of y duration in the future. The task relation generalizes universal laws which are not always possible to articulate for a dynamical system that is too far from deterministic, or otherwise unknown to admit uniform relationships, describing instead the limits of what is dynamically possible.[36]

Letting a nonempty family of sets $\mathcal{S}$ be *directed* just in case $S \subseteq S_1 \cap S_2$ for some $S \in \mathcal{S}$ whenever $S_1, S_2 \in \mathcal{S}$, I will define a *task frame* to be any $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ where W is a nonempty set of world states, $\mathcal{D}$ is a temporal order, and $\Rightarrow$ is a task relation satisfying the following for positive durations $x, y \geqslant 0$:

Compositionality: $w \Rightarrow_{x+y} v$ if and only if $w \Rightarrow_x u$ and $u \Rightarrow_y v$ for some $u \in W$.

Seriality: $w \Rightarrow_x u$ and $v \Rightarrow_x w$ for some $u, v \in W$.

Limit: $\bigcap_{x>0} (w)_x = \{w\}$.

Spherical: $\bigcap \mathcal{S} \neq \varnothing$ for any directed family $\mathcal{S}$ of nonempty fibers and segments.

The frame constraints above require tasks to compose through intermediate world states, where no world state is a dead end in either direction, distinct world states are instantaneously separated, and no world state is missing wherever compatible constraints converge.[37] For any nonempty set of durations $X \subseteq D$, a *partial history* is a function $\tau : X \to W$ where $\tau(x) \Rightarrow_{y-x} \tau(y)$ for all $x, y \in X$. Consider extending a partial history τ by assigning a world state $u \in W$ to a further duration $z \notin X$. Each

[36]Rather than merely recording which transitions are possible, a *stochastic task function* assigns to each world state $w \in W$ and duration $x \geqslant 0$ with $\texttt{fib}(w, x) \neq \varnothing$ a probability distribution $\Phi_x(w, \cdot)$ over the fiber $\texttt{fib}(w, x)$— summing to 1 when the fiber is countable, and a probability measure in the general case— where $\Phi_x(w, u) > 0$ only if $w \Rightarrow_x u$. A stochastic analogue of the *Compositionality* constraint below would require $\Phi_{x+y}(w, v)$ to agree with the composition of $\Phi_x(w, \cdot)$ and $\Phi_y(\cdot, v)$ through the intermediate world states in the manner of the Chapman–Kolmogorov equation, though I will not develop this extension here. The stochastic task function may be given an objective or epistemic reading depending on the application.

[37]The set of *basic opens* $B_{\mathcal{F}} \coloneqq \{(w)_x : w \in W, x \in D,\ \text{and}\ x > 0\}$ generates a topology $\mathcal{T}_{\mathcal{F}} \coloneqq \langle W, \mathcal{O}_{\mathcal{F}} \rangle$ where the set of *open sets* $\mathcal{O}_{\mathcal{F}}$ is the result of closing $B_{\mathcal{F}}$ under arbitrary union and finite intersection. The *Appendix* proves in **T10** that $\mathcal{T}_{\mathcal{F}}$ is *T1*, and hence *R0* as shown in **C3**.

assignment $\tau(x) = w$ for $x \in X$ requires $u \in \texttt{fib}(w, z - x)$ for any candidate u. As **L8** shows in the *Appendix*, fibers imposed from the same side of z nest by *Compositionality*, and so nearer assignments subsume the constraints imposed from farther away. When a nearest past assignment $\tau(x) = w$ and a nearest future assignment $\tau(y) = v$ flank z, the candidates are confined to the segment $[w, v]^{y-z}_{z-x}$ where segments nest toward z. Nevertheless, X may contain no assignment nearest to z on a given side— as when z is a limit of times in X— and so the nested constraints imposed on z may not have a least member. However, *Seriality* and *Compositionality* ensure that every fiber and segment imposed on z is nonempty as shown in **L9**, where **L10** shows that they form a directed family of nonempty sets. *Spherical* then secures a common candidate, and so no further time obstructs a partial history so that every partial history extends to any duration $z \notin X$ as shown in **L13**.

Task frames provide what are also called *non-deterministic dynamical systems*, a special case of labeled transition systems (LTS) in which transitions between world states are labeled by durations.[38] Setting these connections aside until **§4.2**, a *world history* is any partial history $\tau : X \to W$ whose domain X is convex.[39] A world history thereby settles which world state occurs at each time throughout a convex stretch of times, leaving any times before or after that stretch unsettled. For instance, drawing on $\mathbb{Z}$ to provide the space of durations, an individual game of chess may be identified with a world history beginning with the initial board state and ending after a finite number of moves. A world history is *total* just in case its domain is all of D.[40] Since the *Appendix* proves in **T11** that every partial history extends to a total world history, every bounded world history may be identified with a convex fragment of a total world history. By extending the one-point partial history given by the zero-duration loop $w \Rightarrow_0 w$ at each world state, **C5** shows that every world state occurs at some time in some total world history, excluding idle world states.

Although the durations in D form a total order so that a world history passes through each time in its domain exactly once, world histories may assign multiple durations to the same world state. By permitting world states to occur at more than one time in a world history, there may not be a single answer to the question which world states came before or after another world state in a given world history. Rather, it is by specifying a time x in a world history τ that we may determine which world states occur before or after x in τ. In the case of a meandering end game in chess, there is nothing to prevent the chessboard from occupying the same board state more than once. In contrast with the strictly ordered world states of §2.1, nothing prevents the same world state from occurring at many times in a single world history so that $\tau(x) = \tau(y)$ for $x \neq y$. Similarly, nothing prevents world histories from occupying world states in different orders such as chess games transpose move order.

Instead of positing an abundance of primitive time-shifted possible worlds, the same semantic primitives included in a frame $\mathcal{F}$ generate an abundance of world histories.

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[38] A labeled transition system is a triple $\langle S, \Lambda, \to\rangle$ consisting of a set S of states, a set Λ of labels, and a transition relation $s \xrightarrow{a} s'$ where $s, s' \in S$ and $a \in \Lambda$. The relation need not be functional, since a state may have several a-labeled successors or none, which is exactly what allows the task relation $\Rightarrow_x$ to carry indeterminism. See Baier and Katoen [34] for a comprehensive treatment.

[39] A subset $X \subseteq D$ is *convex* just in case $y \in X$ whenever $x, z \in X$ and $x < y < z$.

[40] Every nontrivial ordered group is infinite and unbounded: given $d > 0$, translation invariance yields the strictly ascending chain $0 < d < 2d < 3d < \cdots$, and inverses give $\cdots < -3d < -2d < -d < 0$.

Writing $H_{\mathcal{F}}$ for the set of all total world histories defined over the frame $\mathcal{F}$, **C5** proves $H_{\mathcal{F}}$ is nonempty. Since $\mathcal{D}$ is translation invariant, we may define the following for any total world histories $\tau, \sigma \in H_{\mathcal{F}}$ and durations $x, y, d \in D$:

Translation: $\bar{a}(z) \coloneqq z + d$ for some $d \in D$.

Time-Shift: $\tau \approx_x^y \sigma$ *iff* there is a translation $\bar{a}$ where $y = \bar{a}(x)$ and $\tau(z) = \sigma(\bar{a}(z))$.

Possible World: $[\tau]_{\mathcal{F}} \coloneqq \{\sigma \in H_{\mathcal{F}} \mid \tau \approx_x^y \sigma$ for some $x, y \in D\}$.

By taking the equivalence class $[\tau]_{\mathcal{F}}$ of world histories time-shifted from τ to represent a *possible world*, one might define $\mathbb{W}_{\mathcal{F}} \coloneqq \{[\tau]_{\mathcal{F}} \mid \tau \in H_{\mathcal{F}}\}$ to be the set of all possible worlds for the frame $\mathcal{F}$. Whereas $\mathbb{W}_{\mathcal{F}}$ encodes the distinct paths through the space of all world states W, the world histories parameterize those paths by assigning times to world states while acknowledging that the choice of origin is arbitrary. Although one might take $\mathbb{W}_{\mathcal{F}}$ to hold a metaphysical standing that $H_{\mathcal{F}}$ cannot, it is $H_{\mathcal{F}}$ that will play an important role in the semantics. Since the classes in $\mathbb{W}_{\mathcal{F}}$ will play no further role in the semantics, I will refer to $H_{\mathcal{F}}$ as *possible worlds* for familiarity.

Having presented the construction of possible worlds, a *model* of $\mathcal{L}$ is any tuple $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ where $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ is a task frame and $|p_i| \subseteq W$ for every sentence letter $p_i \in \mathbb{L}$. The well-formed sentences of $\mathcal{L}$ are evaluated at a *possible world* $\tau \in H_{\mathcal{F}}$ and time $x \in D$ included in a model $\mathcal{M}$ by way of the following semantics:

(p_i) $\mathcal{M}, \tau, x \vDash p_i$ *iff* $\tau(x) \in |p_i|$.

($\bot$) $\mathcal{M}, \tau, x \nvDash \bot$.

($\rightarrow$) $\mathcal{M}, \tau, x \vDash \varphi \rightarrow \psi$ *iff* $\mathcal{M}, \tau, x \nvDash \varphi$ or $\mathcal{M}, \tau, x \vDash \psi$.

($\Box$) $\mathcal{M}, \tau, x \vDash \Box\varphi$ *iff* $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in H_{\mathcal{F}}$.

($\boxed{\scriptstyle P}$) $\mathcal{M}, \tau, x \vDash \boxed{\scriptstyle P}\varphi$ *iff* $\mathcal{M}, \tau, y \vDash \varphi$ for all $y \in D$ where $y < x$.

($\boxed{\scriptstyle F}$) $\mathcal{M}, \tau, x \vDash \boxed{\scriptstyle F}\varphi$ *iff* $\mathcal{M}, \tau, y \vDash \varphi$ for all $y \in D$ where $x < y$.

Despite taking possible worlds to be defined rather than primitive points, possible worlds and times play the same conceptual roles that they have traditionally played in bimodal frameworks. Given a possible world together with a time in a model of $\mathcal{L}$, we may determine the truth-value of any sentence in the language. Whereas the semantics for the metaphysical modal $\Box$ quantifies over all possible worlds in $H_{\mathcal{F}}$, the semantics for the tense operators quantifies over all times in D.[41]

Consider a possible world α for the chessboard in which Black nearly checkmates White on move 31 before blundering the dark squared bishop. Playing on until move 47 in α, Black manages to win the end game, saying:

(K) If I hadn't blundered my bishop, I would have won much sooner.

[41] Although it is easy to define the *reflexive tense operators* $\boxed{\scriptstyle P}'\varphi \coloneqq \boxed{\scriptstyle P}\varphi \wedge \varphi$ and $\boxed{\scriptstyle F}'\varphi \coloneqq \boxed{\scriptstyle F}\varphi \wedge \varphi$ in terms of the *irreflexive tense operators* $\boxed{\scriptstyle P}$ and $\boxed{\scriptstyle F}$ given above, the irreflexive tense operators cannot be defined in terms of the reflexive tense operators. The proof requires a bisimulation argument, and will be omitted here.

Although the present framework is not equipped to interpret tensed counterfactual conditionals, it is clear that Black is lamenting the existence of another possible game in which the blunder had no occurred.[42] Nevertheless, we may evaluate the following:

(P) Black could have checkmated White much sooner.

Given a sufficiently strong reading of 'could', we may regiment P in $\mathcal{L}$ as $\Diamond\langle\mathrm{P}\rangle p_w$ where p_w reads 'White is in checkmate'. The sentence $\Diamond\langle\mathrm{P}\rangle p_w$ is true in α at move 47 just in case there is a possible world β where $\langle\mathrm{P}\rangle p_w$ is true at move 47. Although the finite game in which the blunder is avoided may end before move 47, that game is a bounded world history which extends to a possible world $\beta \in H_{\mathcal{F}}$ by **T11**, and so the present framework permits sentences to be evaluated at move 47 in β. For instance, if p_w is true at move 31 in β, then $\langle\mathrm{P}\rangle p_w$ is true in β at move 47.[43]

Having provided a theory of truth for the language $\mathcal{L}$, it remains to provide a theory of logical consequence by which to survey the valid forms of reasoning warranted by the semantics. I will take the definition to assume the following standard form:

Logical Consequence: $\Gamma \vDash \varphi$ *iff* for any model $\mathcal{M}$ of $\mathcal{L}$, possible world $\tau \in H_{\mathcal{F}}$, and time $x \in D$, if $\mathcal{M}, \tau, x \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M}, \tau, x \vDash \varphi$.

A well-formed sentence φ of $\mathcal{L}$ is *valid* just in case $\varnothing \vDash \varphi$, dropping set notation for convenience. In addition to providing an general framework for semantic theorizing, the task semantics validates a simple and strong logic for $\mathcal{L}$ without imposing additional frame constraints. As **C6** shows, the perpetuity principles are valid:

P1 $\Box\varphi \rightarrow \triangle\varphi$.

P2 $\triangledown\varphi \rightarrow \Diamond\varphi$.

Suppose for contradiction that **P1** has a counterinstance, and so for some well-formed sentence φ, it is metaphysically necessary that φ and yet it is sometimes not the case that φ. Put semantically, $\mathcal{M}, \tau, x \vDash \Box\varphi$ and $\mathcal{M}, \tau, x \nvDash \triangle\varphi$, and so $\mathcal{M}, \tau, y \nvDash \varphi$ for some $y \in D$. We may then define the possible world $\sigma(z) = \tau(z - x + y)$ so that $\mathcal{M}, \sigma, x \nvDash \varphi$, where $\sigma \in H_{\mathcal{F}}$ since the possible worlds are close under translation **L14**. Thus it follows that $\mathcal{M}, \tau, x \nvDash \Box\varphi$, contradicting the above. This proves that **P1** is valid where it follows by classical reasoning that **P2** is equivalent.

Instead of admitting countermodels to **P1** and **P2** as in Kaplan's [2] semantics, the present approach validates the perpetuity principles without imposing *ad hoc* model constraints that undermine the significance of the truth-conditions for the language. Not only do all paradigm examples conform to these principles, **P1** and **P2** support an account of metaphysical modality as the strongest objective modality whereby we may insist that φ is not metaphysically necessary if φ ever fails to be the case. Moreover, by defining possible worlds in terms of the world states, tasks, and durations, the present approach maintains the standard semantic clauses for both tense and modal operators

[42] I develop a hyperintensional semantics for tensed counterfactual conditionals in Brast-McKie [6].

[43] Alternatively, one might evaluate sentences at any world history τ— total or not— together with a time $x \in \mathtt{dom}(\tau)$, taking $\Box$ to quantify over all world histories σ where $x \in \mathtt{dom}(\sigma)$, restricting $\boxed{\mathrm{P}}$ and $\boxed{\mathrm{F}}$ to the times in $\mathtt{dom}(\tau)$, and adapting logical consequence to replace D with $\mathtt{dom}(\tau)$. Since evaluating sentences only at possible worlds excludes no course of events by **T11**, the present account retains the stronger tense logic.

without accepting instrumentalism or temporal absolutism about a primitive ontology of time-shifted worlds. In addition to these merits, taking the metaphysical modals to quantify over possible worlds rather than world-time pairs provides a more expressive theory than Montague's [1] semantics. I will provide further abductive support for the task semantics given above by presenting a logic for $\mathcal{L}$ in **§3.2**. The following subsection will begin by extending $\mathcal{L}$ to include a stability operator in order to demonstrate the power of the present approach in contrast to two-dimensional semantic theories.

3.1 Restricted Modalities

The semantics for metaphysical necessity $\Box$ quantifies over all possible worlds in $H_{\mathcal{F}}$ defined over the frame $\mathcal{F}$, thereby validating an S5 modal logic. Although this is in keeping with the interpretation of metaphysical modality as the strongest objective modality, there are applications which call for restricted modals. For instance, given a world $\tau \in H_{\mathcal{F}}$ and time $x \in D$, we may let $\langle \tau \rangle_x := \{\sigma \in H_{\mathcal{F}} \mid \sigma(x) = \tau(x)\}$ be the set of possible worlds that intersect τ at x in order to provide the following semantics:

($\boxdot$) $\mathcal{M}, \tau, x \vDash \boxdot\varphi$ *iff* $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in \langle \tau \rangle_x$.

A sentence $\boxdot\varphi$ is true in a world τ at a time x just in case φ is true at time x in every possible world that occupies the same world state as τ at x.[44] Letting $\Diamonddot\varphi := \neg\boxdot\neg\varphi$, the *stability operators* $\boxdot$ and $\Diamonddot$ may be used to define the following modals:

Will Always: $\boxed{!}\varphi := \boxdot\boxed{\mathrm{F}}\varphi$. *Could Always:* $\boxed{?}\varphi := \Diamonddot\boxed{\mathrm{F}}\varphi$.

Will Eventually: $\Diamond_{!}\varphi := \boxdot\Diamond_{\mathrm{F}}\varphi$. *Could Eventually:* $\Diamond_{?}\varphi := \Diamonddot\Diamond_{\mathrm{F}}\varphi$.

Given move number x in a chess game α, the operators above may be used to quantify over the games of chess which occupy the same board state as the game α at time x. Letting p_w be the sentence 'The white king is in checkmate', a game of chess may be resigned if the white king will eventually be in checkmate: $\Diamond_{!} p_w$. By contrast, letting p_b be the sentence 'The black king is in checkmate', a game of chess is still worth playing if each player could eventually win which we may express as: $\Diamond_{?} p_b \wedge \Diamond_{?} p_w$.

Another natural restriction on the set of possible worlds arises from considering only those possible worlds which overlap with a given world up to the present time while possibly diverging in the future, and similarly for the past. More precisely:

Open Futures: $|\tau\rangle_x := \{\sigma \in H_{\mathcal{F}} \mid \sigma(y) = \tau(y) \text{ for all } y \leqslant x\}$.

Open Pasts: $\langle\tau|_x := \{\sigma \in H_{\mathcal{F}} \mid \sigma(y) = \tau(y) \text{ for all } y \geqslant x\}$.

Whereas $|\tau\rangle_x$ is the set of all possible worlds that occupy the same world state as τ at each time up to and including x while possibly diverging at later times, $\langle\tau|_x$ is the set of all possible worlds that occupy the same world states as τ at x and all later times

[44]Since $\langle \tau \rangle_x$ is an equivalence class under the relation $\sigma \sim_x \tau := \sigma(x) = \tau(x)$, the monomodal logic of $\boxdot$ is also S5. However, for non-temporal φ, the truth-value of φ at $\langle \mathcal{M}, \tau, x \rangle$ depends only on $\tau(x)$, so $\varphi \to \boxdot\varphi$ is valid, collapsing $\boxdot$ to the trivial modality on this fragment.

while possibly diverging at earlier times. Given these definitions, we may introduce operators which quantify over these restricted sets of possible worlds:

($\boxed{\triangleright}$) $\mathcal{M}, \tau, x \vDash \boxed{\triangleright}\varphi$ *iff* $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in |\tau\rangle_x$.

($\boxed{\triangleleft}$) $\mathcal{M}, \tau, x \vDash \boxed{\triangleleft}\varphi$ *iff* $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in \langle\tau|_x$.

Whereas the *open future* operator $\boxed{\triangleright}$ quantifies over all possible worlds that agree with the world of evaluation up to the time of evaluation, the *open past* operator $\boxed{\triangleleft}$ is also intelligible and has been included for comparison. Although there is much greater occasion to contemplate the range of open futures at a moment while making plans, there is nothing to stop us from also quantifying over the range of open pasts.

Given that the sets of possible worlds $\langle\tau\rangle_x$, $|\tau\rangle_x$, and $\langle\tau|_x$ are definable in terms of the construction of possible worlds, there is no need to posit additional primitive accessibility relations between possible worlds in order to provide a semantics for the intersection, open future, and open past operators. By contrast, taking possible worlds to be structureless points requires each frame to include primitive accessibility relations $R_\times$, $R_\triangleright$, and $R_\triangleleft$ in order to identify the appropriate subsets of possible worlds for these operators to quantify over. Since not all accessibility relations provide appropriate restrictions on the space of possible worlds, a number of further frame constraints would have to be imposed if $\boxdot$, $\boxed{\triangleright}$, and $\boxed{\triangleleft}$ are to maintain their intended readings. At the very least, we ought to expect $R_\triangleright$ and $R_\triangleleft$ to be restrictions of $R_\times$ where the intersection of $R_\triangleright$ and $R_\triangleleft$ is nonempty. Even so, merely imposing a range of constraints on the accessibility relations in accordance with the intended readings of the operators does not uniquely determine their extensions. Although permissible, these theoretical costs are avoided entirely by the present theory. Instead of positing a range of frame constraints, the construction of possible worlds makes it provable that $|\tau\rangle_x \subseteq \langle\tau\rangle_x$ and $\langle\tau|_x \subseteq \langle\tau\rangle_x$ where $|\tau\rangle_x \cap \langle\tau|_x = \{\tau\}$. Rather than positing primitive accessibility relations together with a range of constraints, for each fixed duration $x \in D$ we may define $R_\times(\tau, \sigma) \coloneqq \sigma \in \langle\tau\rangle_x$, $R_\triangleright(\tau, \sigma) \coloneqq \sigma \in |\tau\rangle_x$, and $R_\triangleleft(\tau, \sigma) \coloneqq \sigma \in \langle\tau|_x$, avoiding the need to impose frame constraints which align with an intended interpretation. Since $R_\times$, $R_\triangleright$, and $R_\triangleleft$ are definable, we may omit the extra notation entirely, further indicating the power and flexibility of the present approach.

Rather than accepting the costs of imposing constraints on a primitive accessibility relation between possible worlds, we may extend the present framework to take the *task relation* to be four-place so that $u \Rightarrow_x^w v$ indicates that it is possible for the world state u to transition to the world state v in duration x relative to the world state w. This extension accommodates laws that are contingent over the frame or even temporary within a given world history. For instance, we may consider a game where the rules may change during the course of game. Letting $H_\mathcal{F}^w$ be the set of total world histories defined over $\Rightarrow^w$, i.e., the total functions $\tau : D \to W$ where $\tau(x) \Rightarrow_{y-x}^w \tau(y)$ for all $x, y \in D$, we may provide the following semantic clause for nomological necessity:

($\boxed{\mathrm{N}}$) $\mathcal{M}, \tau, x \vDash \boxed{\mathrm{N}}\varphi$ *iff* $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in H_\mathcal{F}^{\tau(x)}$.

Without assuming $\Rightarrow^w$ to be invariant with respect to the world states $w \in W$, nothing requires $\boxed{\mathrm{N}}$ to have an S5 logic. By contrast, I will take metaphysical necessity $\Box$ to be the strongest objective modality where an S5 logic is appropriate.[45] Since the present aim is to develop a bimodal logic for tense and metaphysical modality, I will omit further consideration of the restricted modals $\boxdot$, $\boxed{\triangleright}$, $\boxed{\triangleleft}$, and $\boxed{\mathrm{N}}$.

3.2 Bimodal Logic

Recall the propositional language $\mathcal{L} = \langle \mathbb{L}, \bot, \to, \Box, \boxed{\mathrm{P}}, \boxed{\mathrm{F}} \rangle$ where $\mathbb{L} \coloneqq \{p_i \mid i \in \mathbb{N}\}$ is a set of *sentence letters* and the *well-formed sentences* of $\mathcal{L}$ are defined as follows:

$$\varphi,\ \psi \ ::= \ p_i \mid \bot \mid \varphi \to \psi \mid \Box\varphi \mid \boxed{\mathrm{P}}\varphi \mid \boxed{\mathrm{F}}\varphi.$$

Letting $\varphi_{\langle \mathrm{P}|\mathrm{F} \rangle}$ be the result of exchanging all occurrences of $\boxed{\mathrm{P}}$ and $\boxed{\mathrm{F}}$ in φ, the *Logic of Tense and Modality* **TM** is the smallest extension of *Classical Propositional Logic* **CPL** to be closed under all instances of the following axiom and rule schemata:

MP $\varphi, \varphi \to \psi \vdash \psi$.

MN *If* $\vdash \varphi$, *then* $\vdash \Box\varphi$.

MK $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.

MT $\Box\varphi \to \varphi$.

M5 $\Diamond\Box\varphi \to \Box\varphi$.

MF $\Box\varphi \to \Box\boxed{\mathrm{F}}\varphi$.

TD *If* $\vdash \varphi$, *then* $\vdash \varphi_{\langle \mathrm{P}|\mathrm{F} \rangle}$.

TK $\boxed{\mathrm{F}}(\varphi \to \psi) \to (\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\psi)$.

T4 $\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\boxed{\mathrm{F}}\varphi$.

TB $\Diamond_{\mathrm{F}}\top$.

TA $\varphi \to \boxed{\mathrm{F}}\Diamond_{\mathrm{P}}\varphi$.

TL $(\Diamond_{\mathrm{F}}\varphi \wedge \Diamond_{\mathrm{F}}\psi) \to [\Diamond_{\mathrm{F}}(\Diamond_{\mathrm{F}}\varphi \wedge \psi) \vee \Diamond_{\mathrm{F}}(\varphi \wedge \psi) \vee \Diamond_{\mathrm{F}}(\varphi \wedge \Diamond_{\mathrm{F}}\psi)]$.

The soundness of **TM** and its extensions is sketched in **§5.4** of the *Appendix*, where **C7** presents the completeness results for the $\mathcal{L}^+$ systems.[46] Whereas **MP**, **MN**, **MK**, **MT**, and **M5** are familiar from modal logic, **TD** makes the logic symmetric with respect to the past and future at each time and **TK** distributes the future operator over material implication. Additionally, **T4** requires the temporal ordering to be transitive and **TB** asserts that every time has a strictly later time, guaranteeing seriality where the past dual follows by **TD**. Whereas **TA** asserts that the present is past to all future times, **TL** asserts that any two future times are linearly ordered. Given **TD**, we may also conclude that the present is future to all past times and any two past times are linearly ordered from which it follows that time is linear.

Whereas the axioms and rules discussed so far include either modal or temporal operators, **MF** is the sole bimodal interaction axiom asserting that what is necessary is necessarily always going to be the case. Since $\Box\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\varphi$ is an instance of **MT**, $\Box\varphi \to \boxed{\mathrm{F}}\varphi$ follows from **MF** where $\Box\varphi \to \boxed{\mathrm{P}}\varphi$ follows by **TD**. Together with **MT**, these results entail $\Box\varphi \to (\boxed{\mathrm{P}}\varphi \wedge \varphi \wedge \boxed{\mathrm{F}}\varphi)$ which is equivalent to the perpetuity principles:

[45] The account of the strongest objective modality in **§5.1** derives the **4** and **B** axioms in **C2** where **T**, **K**, and necessitation are built into the characterization. Thus an **S5** logic for $\Box$ is included in **TM** below.

[46] See https://github.com/benbrastmckie/BimodalLogic for the Lean 4 formalization.

P1 $\Box\varphi \to \triangle\varphi$. **P2** $\triangledown\varphi \to \Diamond\varphi$.

This derivation shows that the perpetuity principles follow from **MF** and **MT** by classical reasoning. In addition to being stated in primitive terms, **MF** and **MT** are easy to justify. We may also derive the following principle:

TF $\Box\varphi \to \boxed{\scriptstyle F}\Box\varphi$.

Whereas $\Box\varphi \to \Box\Box\varphi$ follows by standard modal reasoning, $\Box\Box\varphi \to \Box\boxed{\scriptstyle F}\Box\varphi$ and $\Box\boxed{\scriptstyle F}\Box\varphi \to \boxed{\scriptstyle F}\Box\varphi$ are instances of **MF** and **MT**. Composing yields $\Box\varphi \to \boxed{\scriptstyle F}\Box\varphi$ where $\Box\varphi \to \boxed{\scriptstyle P}\Box\varphi$ follows by **TD**. We may then derive $\Box\varphi \to (\Box\boxed{\scriptstyle P}\varphi \wedge \Box\varphi \wedge \Box\boxed{\scriptstyle F}\varphi)$ where $\Box\varphi \to \Box(\boxed{\scriptstyle P}\varphi \wedge \varphi \wedge \boxed{\scriptstyle F}\varphi)$ follows by modal reasoning. Given the definitions, we have:

P3 $\Box\varphi \to \Box\triangle\varphi$. **P4** $\Diamond\triangledown\varphi \to \Diamond\varphi$.

Since $\Box\Diamond\varphi \to \Diamond\varphi$ is an instance of **MT**, it follows from **TK** that $\boxed{\scriptstyle F}\Box\Diamond\varphi \to \boxed{\scriptstyle F}\Diamond\varphi$. We may then derive $\Diamond\varphi \to \Box\Diamond\varphi$ from **M5**, where $\Box\Diamond\varphi \to \boxed{\scriptstyle F}\Box\Diamond\varphi$ is an instance of **TF**. Thus $\Diamond\varphi \to \boxed{\scriptstyle F}\Diamond\varphi$ where $\Diamond\varphi \to \boxed{\scriptstyle P}\Diamond\varphi$ follows by **TD**, and so $\Diamond\varphi \to (\boxed{\scriptstyle P}\Diamond\varphi \wedge \Diamond\varphi \wedge \boxed{\scriptstyle F}\Diamond\varphi)$ which is equivalent to $\Diamond\varphi \to \triangle\Diamond\varphi$. Given **P4**, we may derive the following:

P5 $\Diamond\triangledown\varphi \to \triangle\Diamond\varphi$. **P6** $\triangledown\Box\varphi \to \Box\triangle\varphi$.

The perpetuity principles **P1** – **P6** begin to characterize the interactions between tense and modality in **TM**. A number of additional theorems will be derived in **§5** where the remainder of the present section will strengthen **TM** by imposing additional frame constraints in addition to extending the expressive power of the language.

3.3 Extensions

Since the logic for metaphysical modality already describes the strongest objective modality, it remains to strengthen the tense logic by constraining the temporal order $\mathcal{D}$. In addition to taking $\mathcal{D}$ to be a nontrivial totally ordered abelian group, consider:

> Discrete: For any time $x \in D$, if there is a later time $y > x$, then there is a least later time $y' > x$ where $z \geqslant y'$ for all $z > x$.[47]
>
> Dense: For any times $x, y \in D$ where $x < y$, there is a time $z \in D$ where $x < z < y$.
>
> Complete: Every nonempty set $X \subseteq D$ bounded above has a least upper bound.

The frame constraints above place restrictions on $\mathcal{D} = \langle D, +, 0, \leqslant\rangle$ which the following axioms characterize as established in **T12**, **T13**, and **T14**, respectively:

DF $(\boxed{\scriptstyle P}\varphi \wedge \varphi \wedge \Diamond_{\scriptstyle F}\top) \to \Diamond_{\scriptstyle F}\boxed{\scriptstyle P}\varphi$.

[47] It follows that if there is an earlier time $y < x$, then there is a greatest earlier time. Since $\mathcal{D}$ is an ordered group, the map $x \mapsto -x$ reverses the order. Given $y < x$, we have $-x < -y$. The Discrete condition provides an earliest time $z > -x$, and so $-z < x$ is the latest time before x.

DN $\boxed{\mathrm{F}}\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\varphi$.[48]

CO $\triangle(\boxed{\mathrm{P}}\varphi \to \Diamond_{\mathrm{F}}\boxed{\mathrm{P}}\varphi) \to (\boxed{\mathrm{P}}\varphi \to \boxed{\mathrm{F}}\varphi)$.

Since no temporal order is both discrete and dense, **TM** cannot be extended to include both **DF** and **DN** while maintaining consistency. Letting $\mathbf{TM}_{\mathrm{F}}$ extend **TM** to include all instances of **DF**, I will take $\mathbf{TM}_{\mathrm{D}}$ to include all instances of **DN**, and $\mathbf{TM}_{\mathrm{C}}$ to include all instances of **CO**, where $\mathbf{TM}_{\mathrm{DC}}$ is the minimal extension of $\mathbf{TM}_{\mathrm{D}}$ and $\mathbf{TM}_{\mathrm{C}}$, corresponding to the continuous temporal orders that are both dense and Dedekind complete.[49] I will assume with **TD** that the past and future have the same logic.

For any task frame $\mathcal{F}$, the semantic clauses for the metaphysical modals quantify over all possible worlds in $H_{\mathcal{F}}$, ensuring that $\Box$ has an **S5** logic as shown in **T21**. Since every possible world is defined over the temporal order $\mathcal{D}$ provided by the frame, the structure that the durations have— discrete or dense, and either Dedekind complete or not— hold of metaphysical necessity for that system. For instance, if $\mathcal{D}$ is dense, then **DN** and its necessitation $\Box(\boxed{\mathrm{F}}\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\varphi)$ are valid over this frame. That density is necessary if true is not a special feature of the task semantics but an instance of the fact that frame validity is closed under necessitation: whatever holds at every point of every model on a frame is necessary there, since $\Box$ quantifies only over that frame's possible worlds. Kripke semantics supplies the precedent. For instance, over the class of symmetric frames, the **B** axiom and its necessitation are both valid, and so symmetry is necessary if true. Structural disputes about metaphysical modality already take this form: whether an **S4** or **S5** logic is correct for $\Box$ is debated as a question of which logic and class of frames are correct, never as a claim that symmetry is itself metaphysically contingent.[50] The correspondence-theoretic methodology assumed in §**2.3** takes a frame constraint paired with a proved correspondence theorem to be the mark of carefully executed semantics, not a liability calling for repair. Since possible worlds are only ever defined over a single frame, there is no modality whose semantics quantifies over possible worlds defined on distinct frames. Disputes about the structure of time concern which class of frames and logic is appropriate for the dynamical systems under study, not what is metaphysically contingent *simpliciter*.[51] It is an advantage of the present framework that such disputes can be articulated in $\mathcal{L}$ without further

[48] Density cannot be expressed in terms of the reflexive tense operators $\boxed{\mathrm{P}}'$ and $\boxed{\mathrm{F}}'$ from footnote 41.

[49] A totally ordered group is *Archimedean* if for any $x, y \in D$ where $x > 0$, there is a positive integer n where $nx > y$. By Hölder's theorem, every Archimedean totally ordered group is abelian, and a nontrivial discrete Archimedean totally ordered abelian group is isomorphic to $\mathbb{Z}$. Since $\mathbb{Z}$ is Dedekind complete, **CO** is valid over all Archimedean discrete frames.

[50] The *Appendix* **5.1** takes the objective modalities to be closed under converses to derive **C2**, providing grounds for thinking that it is an analytic truth that metaphysical modality has an **S5** logic.

[51] Dorr and Goodman [29, p. 656] express sympathy for an account of metaphysical modality able to express theses about the contingency of the structure of time. Within the present framework one might give voice to such contingency by relaxing totality, admitting *irregular worlds*— functions $\tau : X \to W$ where $X \subsetneq D$ is a *coset domain*, a translate $G + c$ of a nontrivial subgroup $G \leqslant \mathcal{D}$, and $\tau(x) \Rightarrow_{y-x} \tau(y)$ for all $x, y \in X$— and defining consequence over the irregular and possible worlds alike. Cosets rather than subgroups, since a family of translates is closed under ambient translation and so preserves **MF** and the perpetuity principles, which the subgroup formulation loses. The price is exact: every nontrivial ordered abelian group contains a discrete cyclic subgroup, so **DN** is then valid over no frame whatever, and **DF** fails over discrete orders possessing a subgroup that is itself dense, such as $\mathbb{Q} \times_{\mathrm{lex}} \mathbb{Z}$, so the correspondence results of **T12**, **T13**, and **T14** collapse together. These considerations recommend possible over irregular worlds.

expressive resources, where **TM**, $\mathbf{TM}^+_{\mathrm{F}}$, $\mathbf{TM}^+_{\mathrm{D}}$, and $\mathbf{TM}^+_{\mathrm{C}}$ completely axiomatize the base, discrete, dense, and Dedekind complete theories of time **C7**.[52]

Besides the various extensions of **TM** that one may consider, there are a number of further operators by which to extend the expressive power of $\mathcal{L}$. In particular, the *next* $\textcircled{\scriptsize F}$ and *previous* $\textcircled{\scriptsize P}$ operators have natural applications given a restriction to the discrete frames over which **DF** and its past dual are valid. Consider the semantics:

($\textcircled{\scriptsize F}$) $\mathcal{M}, \tau, x \vDash \textcircled{\scriptsize F}\varphi$ *iff* $\mathcal{M}, \tau, y \vDash \varphi$ for some $y > x$ where $y \leqslant z$ for all times $z > x$.

($\textcircled{\scriptsize P}$) $\mathcal{M}, \tau, x \vDash \textcircled{\scriptsize P}\varphi$ *iff* $\mathcal{M}, \tau, y \vDash \varphi$ for some $y < x$ where $y \geqslant z$ for all times $z < x$.

Citing Dana Scott, Prior [21, p. 66] introduces operators for *tomorrow* and *yesterday* which are represented here as $\textcircled{\scriptsize F}$ and $\textcircled{\scriptsize P}$ though he does not provide a semantics. Instead, Prior focuses on the *metric tense operators* $Pn\alpha$ and $Fn\alpha$ which indicate that α occurs at a distance n from the time of evaluation in either the past or future. To incorporate metric tense operators, $\mathcal{L}$ would need to include singular terms for durations which I will not explore here. By contrast, semantic clauses for the *since* $\lhd$ and *until* $\rhd$ operators that Kamp [35] introduced can be provided as follows:

($\lhd$) $\mathcal{M}, \tau, x \vDash \varphi \lhd \psi$ *iff* $\mathcal{M}, \tau, z \vDash \psi$ for some time $z < x$ where $\mathcal{M}, \tau, y \vDash \varphi$ for all intermediate times $y \in D$ where $z < y < x$.

($\rhd$) $\mathcal{M}, \tau, x \vDash \varphi \rhd \psi$ *iff* $\mathcal{M}, \tau, z \vDash \psi$ for some time $z > x$ where $\mathcal{M}, \tau, y \vDash \varphi$ for all intermediate times $y \in D$ where $x < y < z$.

We may define $\Diamond_{\mathrm{P}}\varphi \coloneqq \top \lhd \varphi$ and $\Diamond_{\mathrm{F}}\varphi \coloneqq \top \rhd \varphi$, where $\boxed{\scriptsize P}\varphi \coloneqq \neg\Diamond_{\mathrm{P}}\neg\varphi$ and $\boxed{\scriptsize F}\varphi \coloneqq \neg\Diamond_{\mathrm{F}}\neg\varphi$, and so adding $\lhd$ and $\rhd$ to the language obviates the need to take $\boxed{\scriptsize P}$ and $\boxed{\scriptsize F}$ to be primitive.[53] Since there is no intermediate time at which $\bot$ holds, we may also define $\textcircled{\scriptsize F}\varphi \coloneqq \bot \rhd \varphi$ and $\textcircled{\scriptsize P}\varphi \coloneqq \bot \lhd \varphi$ which require the argument to be true at the successor or predecessor, respectively. In case there is no immediate successor or predecessor, $\textcircled{\scriptsize F}$ and $\textcircled{\scriptsize P}$ are equivalent to $\bot$. Whereas the operators $\lhd$ and $\rhd$ have wide ranging applications that presume nothing of the structure of time, $\textcircled{\scriptsize F}$ and $\textcircled{\scriptsize P}$ are only worth including in a language used to study discrete systems. Nevertheless, there are many discrete systems for which $\textcircled{\scriptsize F}$ and $\textcircled{\scriptsize P}$ are natural to consider. For instance, one might consider the moves in a chess game, clock cycles on a chip, sequences of program executions, or the actions of one or more agents in a multi-agent system.

Introducing the operators $\lhd$ and $\rhd$ whose semantics requires the antecedent guard φ to hold for all times since or until the consequent event ψ affords greater expressive power to the language than $\boxed{\scriptsize P}$ and $\boxed{\scriptsize F}$ provide. The discrete, dense, and complete extensions of a base proof system $\mathbf{TM}^+$ for $\mathcal{L}^+ \coloneqq \langle \mathbb{L}, \bot, \rightarrow, \Box, \lhd, \rhd \rangle$ are provided in **§5.5**, where $\mathbf{TM}^+$ extends $\mathcal{L}$ in expressive power and **C7** establishes completeness for these $\mathcal{L}^+$ systems. The soundness and completeness of $\mathbf{TM}^+$ and its extensions are verified in the Lean 4 repository for this paper. Even so, these systems lack the means

[52]Over the unrestricted class of task frames, **DF**, **DN**, and **CO** are not valid, nor are their negations: it is not *analytic*— encoded as *truth in all models*— that time is discrete, dense, or complete, nor that it fails to be. By contrast, **TL** and its necessitation are valid over the class of dense frames, and so it is analytic that time is linear in the present framework in much the same way that identity is reflexive.

[53]The *strict semantics* given here is analogous to the *irreflexive* semantics for $\boxed{\scriptsize P}$ and $\boxed{\scriptsize F}$. See footnote 41.

by which to cross reference either times or worlds. To lend greater discriminating power to the language, we may include a vector $\vec{v} = \langle v_1, v_2, \ldots \rangle$ of stored times and vector $\vec{\mu} = \langle \mu_1, \mu_2, \ldots \rangle$ of stored worlds in the point of evaluation to provide the semantic clauses for the *store operators* $\upharpoonleft^i_{\mathrm{T}}$ and $\upharpoonleft^i_{\mathrm{M}}$ and *recall operator* $\downharpoonleft^i_{\mathrm{T}}$ and $\downharpoonleft^i_{\mathrm{M}}$ for each $i \in \mathbb{N}$:

($\upharpoonleft_{\mathrm{T}}$) $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu} \vDash \upharpoonleft^i_{\mathrm{T}}\varphi$ *iff* $\mathcal{M}, \tau, x, \vec{v}_{[x/v_i]}, \vec{\mu} \vDash \varphi$.

($\downharpoonleft_{\mathrm{T}}$) $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu} \vDash \downharpoonleft^i_{\mathrm{T}}\varphi$ *iff* $\mathcal{M}, \tau, v_i, \vec{v}, \vec{\mu} \vDash \varphi$.

($\upharpoonleft_{\mathrm{M}}$) $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu} \vDash \upharpoonleft^i_{\mathrm{M}}\varphi$ *iff* $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu}_{[\tau/\mu_i]} \vDash \varphi$.

($\downharpoonleft_{\mathrm{M}}$) $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu} \vDash \downharpoonleft^i_{\mathrm{M}}\varphi$ *iff* $\mathcal{M}, \mu_i, x, \vec{v}, \vec{\mu} \vDash \varphi$.

Whereas $\upharpoonleft^i_{\mathrm{T}}\varphi$ replaces the i^{th} value of $\vec{v}$ with the evaluation time and $\upharpoonleft^i_{\mathrm{M}}\varphi$ replaces the i^{th} value of $\vec{\mu}$ with the evaluation world, $\downharpoonleft^i_{\mathrm{T}}\varphi$ shifts the evaluation time to the i^{th} value stored in $\vec{v}$ and $\downharpoonleft^i_{\mathrm{M}}\varphi$ shifts the evaluation world to the i^{th} value stored in $\vec{\mu}$. These operators generalize the *now* and *then* operators that Vlach [36] introduces to resolve tense anaphora, replacing Vlach's single pair of operators with an indexed family $\upharpoonleft^i_{\mathrm{T}}$ and $\downharpoonleft^i_{\mathrm{T}}$ as well as providing their world-theoretic counterparts $\upharpoonleft^i_{\mathrm{M}}$ and $\downharpoonleft^i_{\mathrm{M}}$ to store and recall arbitrarily many times and worlds. Despite adding $\vec{v}$ and $\vec{\mu}$ as parameters to the point of evaluation, the semantics for $\mathcal{L}$ may otherwise be maintained, letting $\mathcal{L}^\star \coloneqq \langle \mathbb{L}, \bot, \rightarrow, \Box, \lhd, \rhd, \boxdot, \upharpoonleft^i_{\mathrm{T}}, \downharpoonleft^i_{\mathrm{T}}, \upharpoonleft^i_{\mathrm{M}}, \downharpoonleft^i_{\mathrm{M}} \rangle$. Although extending **TM** to provide a logic for $\mathcal{L}^\star$ is outside the scope of the present paper, the following section will employ the semantics for $\mathcal{L}^\star$ to address the paradox of the open future and draw further connections to non-deterministic dynamical systems theory.

4 Tense and Modality

The construction of possible worlds assumes that time is a total order. For any world $\tau \in H_{\mathcal{F}}$ and time $x \in D$ in a model $\mathcal{M}$ of $\mathcal{L}$, there is a determinate past and future relative to x where every well-formed sentence of $\mathcal{L}$ has a unique truth-value in τ at x. However natural it may be to take the past to be determined up to any given moment, it is much more contentious to suppose that each time determines a unique future. Arguments as old as Aristotle have sought to lend credence to the idea that the future differs from the past on account of remaining open to determination by admitting a range of incompatible future alternatives, none of which is singled out as the actual future. Thomason [25] brings this point out as follows:

> [T]he basic issue here seems to be whether or not one is prepared to accept as meaningful the assertion that there is always, whether we know it or not, a single possible future which, from the perspective of a given time will be its actual future. (p. 270)

In considering future contingents such as 'There will be a sea battle tomorrow', many have sought to follow Aristotle in denying that such claims have a truth-value until the future comes about. Although there may be no fact of the matter whether there will be a sea battle tomorrow when it is considered today, we may nevertheless expect it to be settled tomorrow whether there ends up being a sea battle or not.

The construction of possible worlds takes each time in each possible world to have a complete past and future that is determined in every respect. However, evaluating a future contingent claim at a possible world and time does not carry any commitment that the world of evaluation is the actual world whose future is guaranteed to take place. By considering all worlds that intersect the world of evaluation at the time of evaluation, one may leave it open which of those worlds will play out. Just as one may consider the various chess games that may transpire from a given board state, it is also natural to consider the various routes between two points on a map, the different states a computer may transition through, or the different sequences of actions by one or more agents. Despite making evaluations according to one possible world or another when considering the evolution of a system, doing so does not require any of the possible worlds to be the *actual* world that will take place. Even if none of the possible worlds are foretold to be actual, we can say in full confidence what each possible world specifies. The following subsection will defend a deflationary approach of this kind by considering the shortcomings that face the alternatives.

4.1 Open Future

Dynamical systems of all kinds admit an open future. Chess continues to serve as a paradigm case. If each board state determined all future board states, there would only be one game of chess to rehearse, leaving little semblance of a game worth playing. Just as the moves in chess narrow the range of open futures by determining what future board state becomes actual, human actions within non-deterministic systems carry consequences. When I choose soup over salad, I determine the next course of my meal, ruling out the futures in which salad precedes the main course while leaving open futures in which I have dessert as well as those in which I do not.

By formally describing both deterministic and non-deterministic systems, we may ask as a strictly empirical matter whether any given system is deterministic or not. Whereas dynamical systems theory provides resources for modeling both deterministic and non-deterministic systems, **TM** provides logical resources for reasoning about dynamical systems. In particular, consider the following axiom schema:

Determined: $\varphi \to \boxdot\varphi$.

If we happen to inhabit a deterministic universe, then all instances of the schema above are true. For instance, if I am going to visit Ladakh, then I will eventually visit Ladakh, or in symbols: $\Diamond_{\mathrm{F}} p_l \to \boxed{!}\, p_l$. More generally, consider the frame constraint:

DETERMINISTIC: For $w, u, v \in W$ and $x \in D$, if $w \Rightarrow_x u$ and $w \Rightarrow_x v$, then $u = v$.

Whereas the frame constraints given in **§3.3** impose a restriction on $\mathcal{D}$, the constraint above articulates what it is for the task relation $\Rightarrow$ to be deterministic. Although **T15** shows that *Determined* is valid over every deterministic frame, **T19** exhibits a non-deterministic frame over which *Determined* is valid, and so *Determined* does not characterize the deterministic frames that satisfy DETERMINISTIC.

The present approach distinguishes between deterministic and non-deterministic systems independent of temporal structure. Thus **TL** may be retained, maintaining compatibility with the view that it is analytic that time is linear. Within each possible world $\tau \in H_{\mathcal{F}}$, time is a total order where every sentence receives a determinate truth-value at each time in τ. The openness of the future consists not in the existence of incomparable future times but in the absence of any possible world which is designated as the *actual* world: different possible worlds may pass through the same world state, where none is singled out as the one that has and will continue to obtain. This stands in contrast to the proposal to accommodate an open future by dropping **TL** and replacing the linear temporal order with a partial order. The remainder of this section considers the difficulties that branching-time faces and explains how the task semantics achieves a superior result while retaining a linear theory of time.

Instead of requiring the times in each frame to be totally ordered as in **§3**, there is the tradition following Prior [21] and Kripke [20] that weakens the tense logic by dropping **TL** and taking the times to form a connected left-linear partial order so that each time $x \in D$ may admit incomparable futures.[54] Despite admitting branching times, one might nevertheless preserve the definition from **§3** on which a partial history is any function $\tau : X \to W$ where $X \subseteq D$ and $\tau(x) \Rightarrow_{y-x} \tau(y)$ for all $x, y \in X$, requiring X to be convex for world histories and total for possible worlds. This proposal faces the same difficulties that arise for the Peircean semantics considered in **§2.1**. For instance, consider a chess game τ at a time x in which there is a future time $y > x$ where the black king is in checkmate and an incomparable future time $z > x$ where the white king is in checkmate. By the semantics for the future operator $\Diamond_{\mathrm{F}}$, 'White is going to win' and 'Black is going to win' are both true, though this would seem impossible. It is much more natural to assert 'White *could eventually* win' and 'Black *could eventually* win' to indicate that there is some future in which White wins and some future in which Black wins, where these are the sorts of chess games that are still worth playing. Such a theorist might take $\Diamond_{\mathrm{F}}$ to read 'could eventually' which requires φ at some later time in some branch, and take $\boxed{\mathrm{F}}$ to read 'will always' which requires φ at all later times across all branches. Although this reading is more fitting here, a branching-time theory of this kind inherits the expressive limitations of the Peircean semantics. For instance, there is no way to express the density of time, insofar as density $\boxed{\mathrm{F}}\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\varphi$ concerns the structure of any one branch rather than all future times distributed over all branches. Following the Ockhamist by introducing a further parameter at which to evaluate sentences that specifies the particular branch within a possible world returns the situation that we were in before where sentences are evaluated at parameters that suffice to determine a fixed past and future.

Consider a game of chess in which both Black and White have a chance at winning after 14 moves have been played. Given the state of the board, there simply is no actual game with a determinate future that proceeds from the present board state. Rather, we may consider various chess games that proceed from the present board state at move 14. By taking turns choosing which moves to make, the players compete in their abilities to determine which game of chess becomes actual. Rather than building indeterminacy into the temporal structure itself by taking time to be only partially ordered, the task semantics for $\mathcal{L}$ locates indeterminacy in the task relation between world states over a duration. Letting time be a total order, each possible world $\tau \in H_{\mathcal{F}}$

[54] One must also give up **TD** to validate $\Diamond_{\mathrm{F}}\top \to (\boxed{\mathrm{P}}\boxed{\mathrm{F}}\varphi \to \triangle\varphi)$ which requires the past to be unique and then add the result of exchanging $\boxed{\mathrm{P}}$ and $\boxed{\mathrm{F}}$ to recover what is otherwise derivable from **TD**.

represents one possible path through the space of world states, constrained only by the transitions which the task relation permits. Different possible worlds intersecting the same world state represent genuine alternatives about how the system could evolve without requiring multiple incomparable future times. So long as none of these possible worlds are assumed to be actual, no harm comes in allowing each possible world to specify a determinate past and future from any given world state at a time.

Something similar may be said for Aristotle's sea battle. Since some of the possible worlds intersecting the present world state include a sea battle on the following day and other intersecting possible worlds do not, it is contingent whether there is going to be a sea battle. In symbols, one might assert $\Diamond_{?}\neg p_b \wedge \Diamond_{?} p_b$ where p_b expresses that a sea battle is taking place and $\Diamond_{?}\varphi \coloneqq \Diamond_{\cdot}\Diamond_{F}\varphi$ as before, though this is easy to satisfy.[55] Given a possible world in a deterministic frame where the ships first clash at 2pm tomorrow, both $\Diamond_{?}\neg p_b$ and $\Diamond_{?} p_b$ could be true as witnessed by the time 11am while the ships were still yet to meet and 2:30pm just after the battle started. Rather, we may consider:

$$\uparrow^{1}_{\mathrm{T}}\Diamond_{F}\uparrow^{2}_{\mathrm{T}}\downarrow^{1}_{\mathrm{T}}(\Diamond_{\cdot}\downarrow^{2}_{\mathrm{T}}\neg p_b \wedge \Diamond_{\cdot}\downarrow^{2}_{\mathrm{T}} p_b) \tag{Sea}$$

The claim above asserts that there is a future time where p_b is contingent on account of being true in one possible future and false in another. More generally, we may consider the following schema which is invalid given the truth of the claim above:

$$\uparrow^{1}_{\mathrm{T}}\Box_{F}\uparrow^{2}_{\mathrm{T}}\downarrow^{1}_{\mathrm{T}}(\Box_{\cdot}\downarrow^{2}_{\mathrm{T}}\neg\varphi \vee \Box_{\cdot}\downarrow^{2}_{\mathrm{T}}\varphi) \tag{Det}$$

For a given future time, the possible worlds intersecting the present need not agree on φ at that time. In particular, during a day of strategic uncertainty, it may be indeterminate whether there will be a sea battle tomorrow on account of possible worlds that intersect with the present in which there is a sea battle twenty-four hours from now as well as intersecting possible worlds in which there is not.[56]

The claim in Sea regiments the sea-battle claim more strictly than is natural, locating the contingency in question at a single future time. The claim that there will be a sea battle tomorrow does not concern any particular moment— twenty-four hours from now or otherwise— but rather the interval spanning tomorrow's daylight. The operators $\lhd$ and $\rhd$ provide a better regimentation, where we may define:

After: $\mathsf{after}(\varphi, \psi) \coloneqq \neg\varphi \rhd (\varphi \wedge \psi)$.

Before: $\mathsf{before}(\varphi, \psi) \coloneqq \neg\varphi \rhd (\psi \wedge \neg\varphi)$.

Whereas $\mathsf{after}(\varphi, \psi)$ is true in τ at x *iff* φ and ψ are both true at some time $z > x$ where φ fails at every time y for which $x < y < z$, and $\mathsf{before}(\varphi, \psi)$ is true in τ at x *iff* ψ is true and φ is false at some time $z > x$ where φ is false at every time y

[55] Given that $\Diamond_{?}\neg p_b \wedge \Diamond_{?} p_b$ is satisfiable, $\Box_{!}\varphi \vee \Box_{!}\neg\varphi$ is invalid despite retaining **TL** and **TD**. However, invalidity is had too easily, since we need only consider a single world in which there are future times where φ and other future times where $\neg\varphi$. Something similar may be said for the satisfiability of $\Diamond_{¿}\neg p_b \wedge \Diamond_{¿} p_b$.

[56] Carinai and Santorio [37] treat 'will' as a non-quantificational modal governed by a selection function rather than a universal quantifier over worlds. To retain bivalence and reject MacFarlane's [38] relativized truth, the indeterminacy of future contingents is located in facts about a plurality of live alternatives rather than in truth-value gaps. The present proposal is more deflationary still, since no possible world is designated actual, even indeterminately.

for which $x < y < z$.[57] Letting p_r and p_s express that the sun rises and that the sun sets respectively, we may regiment the claim that a sea battle takes place tomorrow as $\mathsf{after}(p_r, \mathsf{before}(p_s, p_b))$, asserting that a sea battle occurs after the next sunrise but before the next sunset. The contingency of the sea battle may then be expressed as:

$$\Diamond\mathsf{after}(p_r, \mathsf{before}(p_s, p_b)) \wedge \Diamond\mathsf{after}(p_r, \neg p_b \rhd p_s) \qquad (\mathtt{Sea}')$$

The claim above requires possible worlds intersecting the present to disagree about whether a sea battle occurs within an event-bounded interval rather than a single stored time. Whereas $\Diamond\varphi$ quantifies over all future times indiscriminately, the guarded quantification articulated above confines the sea battle between the events of sunrise and sunset, exhibiting the expressive power of $\lhd$ and $\rhd$.

As **T20** shows, `Det` is valid over any deterministic frame. By contrast, it is easy to conceive of non-deterministic systems in which `Det` is false where `Sea` requires this to be the case. Given a possible world $\tau \in H_{\mathcal{F}}$, time $x \in D$, and stored times $\vec{v}$ and worlds $\vec{\mu}$ in a model $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ where `Sea` is true, there are at least two possible worlds $\sigma_1, \sigma_2 \in \langle\tau\rangle_x$ where there is a sea battle in σ_1 at time y and there is no sea battle in σ_2 at time y. Considered from the world state $\tau(x) = \sigma_1(x) = \sigma_2(x)$, it would be wrong to claim that σ_1 is the actual world that is destined to be, just as it would be wrong to claim that σ_2 is the actual world. However, when evaluated from $\sigma_1(y)$, there is good reason to take σ_1 to have a clear advantage over σ_2, and *vice versa* when evaluated from $\sigma_2(y)$. After all, a sea battle occurs in σ_1 at time y but not so in σ_2, where this is a major difference that cannot be ignored. Even if σ_1 has a better claim to being actual when we arrive at $\sigma_1(y)$, one cannot claim that σ_1 was actual at time x given the intersecting possible world σ_2 and perhaps many others.

The paradox of the open future may be stated as follows: (1) there is a unique actual world; (2) if there is a unique actual world, then there is no future contingency; (3) it appears that there is future contingency. Articulated in terms of the task semantics, the solution is that there is no *actual world* $\alpha \in H_{\mathcal{F}}$ with a determinate past and future. If there were an actual world, one might include it among the semantic primitives in a frame $\mathcal{F}_{@} = \langle W, \mathcal{D}, \Rightarrow, \alpha\rangle$ where α is taken to at least simulate actuality to provide a semantics for an actuality operator @ as follows:

(@) $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu} \vDash @\varphi$ *iff* $\mathcal{M}, \alpha, x, \vec{v}, \vec{\mu} \vDash \varphi$.

Including an actual world α in a frame raises questions about the commitments of the intended models for the language. By contrast, $\uparrow_{\mathrm{M}}$ and $\downarrow_{\mathrm{M}}$ store and recall worlds without any reference to a globally actual world, avoiding the apparent commitment to an actual future while still providing expressive resources for cross referencing worlds. Instead of positing an actual world α with a complete past and future, a weaker commitment takes each frame $\mathcal{F}_{\#} = \langle W, \mathcal{D}, \Rightarrow, \#\rangle$ to include a distinguished *present* $\# \in W$ representing the world state that currently obtains.[58] Given $\#$ and fixing a time n, we may consider all possible worlds that occupy $\#$ at n without any commitment

[57] We could define $\circledF\psi \coloneqq \mathsf{after}(\top, \psi)$ where after generalizes $\circledF$. Something similar may be said for $\circledP$.

[58] Alternatively, one might designate a *past* that includes the world states that have obtained up to and including the world state that currently obtains, providing a growing block theory of time with Broad [39].

to there being a world that includes an actual future. Although far less committing than $\mathcal{F}_@$, the significance of a frame such as $\mathcal{F}_\#$ is fleeting since what obtains now is only momentary. In contrast to both of these proposals, no world state $\# \in W$ has been designated as present, and no possible world $@ \in H_\mathcal{F}$ has been designated as the actual world. Instead of positing the structure required to say what is true *simpliciter*, the semantics for $\mathcal{L}$ only includes what is needed to evaluate truth relative to one world and time or another where none are distinguished.

Looking out at the Aegean, Aristotle may have wondered whether there will be a sea battle tomorrow or not. Although there are various possible worlds intersecting with the present according to which there is going to be a sea battle tomorrow as well as possible worlds in which there is not, no possible world is designated as the actual world that is already destined to take place. Rather, it remains open which trajectory through the space of world states we may traverse where the future that becomes actual is often determined in part by our influence. By the time tomorrow comes we will know whether there has been a sea battle or not, and so nothing regarding the sea battle will remain open to determination. But arriving where we have by one trajectory or another does not entail that we were always bound to arrive where we have from any point in the past. That we may model our various possible trajectories through the space of world states does not require any trajectory to be actual. At most we may say that the present obtains, though the present is fleeting.

4.2 Dynamical Systems

Dynamical systems theory provides a general mathematical framework for modeling the possible evolutions of a system, suggesting a natural connection with tense and modal reasoning. Whereas the frames defined above take time to have both group and order structure, non-deterministic dynamical systems typically consist of a set of states W, a monoid $\langle D, +, 0\rangle$ for positive durations, and a set of relations $\{R_x\}_{x\in D}$ indexed by D that satisfy the following conditions for all $w, u, v \in W$ and $x, y \in D$:[59]

Nullity: $R_0(w, u)$ if and only if $w = u$.

Compositionality: If $R_x(w, u)$ and $R_y(u, v)$, then $R_{x+y}(w, v)$.

Interpolation: If $R_{x+y}(w, v)$, then $R_x(w, u)$ and $R_y(u, v)$ for some $u \in W$.

The relation $R_x(w, u)$ expresses with an indexed family of two-place relations what $w \Rightarrow_x u$ expresses with a three-place task relation. Every task frame satisfies the conditions above together with *Seriality*, *Limit*, and *Spherical*, where *Nullity* is derived in **L7**. Since a monoid $\langle D, +, 0\rangle$ supplies only positive durations, composition among them is unrestricted. However, to provide semantic clauses for tense operators, an order is also included where every duration x has an inverse $-x$ and the order of addition does not make a difference. Thus **D22** restricts attention to dynamical systems in which the durations for a totally ordered abelian group $\mathcal{D} = \langle D, +, 0, \leqslant\rangle$. Given that

[59] A *monoid* is any $\langle M, \cdot, 1_M\rangle$ where: (1) $a \cdot b \in M$ whenever $a, b \in M$; (2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in M$; and (3) $a \cdot 1_M = 1_M \cdot a = a$ for all $a \in M$.

the task relation induces a canonical topology on world states, **T10** establishes that this topology is *T1* for every frame, with *R0* following as **C3**.

A dynamical system is *deterministic* if R_x is a total function for all $x \in D$, justifying the notation for the *world functions* $\{f_x\}_{x \in D}$ where $f_x(w) = u$ just in case $R_x(w, u)$. A *deterministic model* is any $\mathcal{M}_D = \langle W, \mathcal{D}, \{f_x\}_{x \in D}, |\cdot|\rangle$ where $|p_i| \subseteq W$ for all $p_i \in \mathbb{L}$ and $\langle W, \mathcal{D}, \{f_x\}_{x \in D}\rangle$ is the deterministic dynamical system. Drawing on these resources, Williamson [4, 5] provides a semantics for a higher-order bimodal language in terms of the primitive world functions provided by a deterministic model. I will present the propositional fragment of Williamson's semantics as follows:

(p_i) $\mathcal{M}_D, w \vDash p_i$ *iff* $w \in |p_i|$.

($\bot$) $\mathcal{M}_D, w \nvDash \bot$.

($\rightarrow$) $\mathcal{M}_D, w \vDash \varphi \rightarrow \psi$ *iff* $\mathcal{M}_D, w \nvDash \varphi$ or $\mathcal{M}_D, w \vDash \psi$.

($\Box$) $\mathcal{M}_D, w \vDash \Box\varphi$ *iff* $\mathcal{M}_D, u \vDash \varphi$ for all $u \in W$.

($\boxed{\scriptstyle P}$) $\mathcal{M}_D, w \vDash \boxed{\scriptstyle P}\varphi$ *iff* $\mathcal{M}_D, f_x(w) \vDash \varphi$ for all $x \in D$ where $x < 0$.

($\boxed{\scriptstyle F}$) $\mathcal{M}_D, w \vDash \boxed{\scriptstyle F}\varphi$ *iff* $\mathcal{M}_D, f_x(w) \vDash \varphi$ for all $x \in D$ where $x > 0$.

Instead of interpreting sentences at a model, possible world, and time, the semantic clauses above evaluate sentences directly at a model and world state. This approach may claim the advantage of validating the perpetuity principles **P1** $-$ **P6** from before without undermining the intelligibility of the truth-conditions for the sentences of the language. Nevertheless, the semantics above is restricted to deterministic systems, and so unable to model systems in which states fail to determine a unique past and future. For instance, the board states in a game of chess do not typically mandate any particular continuation, leaving open many different incompatible futures.

Since world functions do not accommodate any degree of future contingency, one might attempt to weaken the temporal order in an attempt to encode an open future. As already observed, taking $\mathcal{D}$ to be a partial order presents the difficulty facing the Peircean semantics which quantifies over all or some incomparable future times. For instance, it is absurd to admit that both Black and White are going to win given incomparable future times in which Black and White each win. Instead of attempting to accommodate the open future, it is natural to restrict applications of Williamson's semantics to deterministic systems where exactly one history intersects each world state. For instance, the motions of the planets are well modeled by a deterministic system constrained under Newton's laws. Given the total state of the solar system including the position and momentum of each planet, all future states of that system are fully determined. If world states determine their past and future, there is no need to specify a temporal parameter alongside a possible world, obviating the need to construct possible worlds. Even so, these are small gains in compensation for giving up the ability to represent non-deterministic systems, especially when metaphysical modality is under discussion. If there are systems whose world states have incompatible futures, the semantics for the strongest objective modality must quantify over all futures for those systems rather than requiring the future to be unique.

The study of dynamical systems is not new. Since the time of Galileo, Newton, and Leibniz during the seventeenth century, it has been standard practice to represent the evolution of a system by using functions. By contrast, it is unheard of not just in physics but throughout the sciences to represent the evolution of a system by a primitive point. The dynamics described by the semantics for languages with tense and modal operators are no different. Thomason [40] presses this point as follows:

> Physics should have helped us to realize that a temporal theory of a phenomenon X is, in general, more than a simple combination of two components: the statics of X and the ordered set of temporal instants. The case in which all functions from times to world states are allowed is uninteresting; there are too many such functions, and the theory has not begun until we have begun to restrict them. [...] The general moral, then, is that we shouldn't expect the theory of time + X to be obtained by mechanically combining the theory of time and the theory of X. (p. 135)

Montague [1] and Kaplan's [2] two-dimensional semantics assumes a dynamics that predates the development of functional dynamics during the Scientific Revolution. Although dynamical systems are nothing new, what has not been adequately explored is the connection between functional dynamics and bimodal reasoning. This paper provides a step in that direction. In the following section, I will conclude by drawing connections to a number of recent developments in logic and computer science.

4.3 Conclusion

Defining possible worlds to be the total world histories $\tau : D \to W$ from durations to world states validates the perpetuity principles **P1** − **P6** while maintaining an expressive semantics for the tense and modal operators. By encoding the full range of non-deterministic dynamical systems, the task semantics accommodates an open future without weakening the temporal order: indeterminacy is located in the diversity of possible worlds passing through a given world state rather than incomparable time following Prior. The soundness theorem **T26** for **TM** and its extensions, together with the completeness results of **C7** for the $\mathcal{L}^+$ systems, implemented in Lean 4, show the task semantics to be correct and adequate for reasoning about non-deterministic dynamical systems. The perpetuity principles are derived from the bimodal axiom **MF** which axiomatizes the free translation of possible worlds.

These results address the deficiencies of extant bimodal logics. Whereas Montague [1] works with a less expressive modality that trivializes the perpetuity principles, Kaplan's [2] semantics invalidates perpetuity. Restricting to the abundant models in **§2.3** restores validity, but only by imposing a model constraint in place of a frame constraint, forfeiting the correspondence between **MF** and the temporal structure. It is this valuation-dependence that forces abundance theorists to posit all merely temporal differences between primitive worlds, and so to embrace temporal absolutism or instrumentalism. Williamson's [4] deterministic semantics validates the perpetuity principles but is restricted to systems in which each world state fixes a unique future, and so is unable to represent non-deterministic systems. By introducing a task relation to constrain transitions between world states over a duration without requiring the task relation to be functional, the task semantics validates the perpetuity principles, handles non-determinism, avoids temporal absolutism, and preserves the expressive power of the semantic clauses for the tense and modal operators.

The task semantics is closely related to several neighboring research programs. Belnap, Perloff, and Xu's [41] agentive `STIT` theory and Rumberg's [27] transition semantics follow Thomason's [25] reconstruction of Prior's [21] Ockhamist semantics. Both frameworks represent possibilities without committing to a fully determined actual future by taking the temporal order to be partial, admitting incomparable moments that branch into alternative futures. By contrast, the task semantics takes the durations $\mathcal{D}$ to be totally ordered, accommodating indeterminism by way of the task relation which unfolds into a branching tree instead of positing the tree outright. Instead of defining the models over a partial order $\langle T, \prec \rangle$, the models of $\mathcal{L}$ are defined over a task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ which is a labeled transition system (LTS) where the transitions are provided by the task relation $w \Rightarrow_x u$ between world states and the labels are durations in $\mathcal{D}$ carrying the group structure. Since the domain of each possible world is a linearly ordered convex set of durations, the framework avoids the difficulties that beset the Peircean future tense operator over branching time outlined in **§2.1**. When W is finite and $D = \mathbb{Z}$, the set of possible worlds $H_{\mathcal{F}}$ is a shift of finite type where world states correspond to alphabet symbols, the unit-step task relation corresponds to the edges of a labeled graph, and possible worlds correspond exactly to the bi-infinite paths through that graph.[60] The general construction extends this correspondence to continuous-time and infinite-state systems. A possible world is a labeled run of this dynamical system. The first-order definability and axiomatizability results that Rumberg and Zanardo [43] establish provide a useful benchmark that the present framework meets: **T12**, **T13**, and **T14** give the corresponding definability results, where the completeness result **C7** for the $\mathcal{L}^+$ systems extends the benchmark to the discrete, dense, and continuous classes. Rather than ensuring axiomatizability by adding a freely chosen bundle of histories as a further component of the models, $H_{\mathcal{F}}$ is the maximal set of possible world determined by a frame $\mathcal{F}$.

The debate in computer science between branching-time and linear-time temporal logics receives a distinctive resolution in the present framework.[61] Individual possible worlds are linearly ordered as in LTL [47], while the set of possible worlds through a given world state has a branching structure as in CTL [48]. The key difference is that $\Box$ quantifies universally over all possible worlds in $H_{\mathcal{F}}$ at the same time rather than employing the state-relativized path quantifiers A and E of CTL. This universal quantification is what validates the perpetuity principles: $\Box\varphi \rightarrow \triangle\varphi$ holds because the time-shift invariance of $H_{\mathcal{F}}$ guarantees that every possible world satisfying φ at one time has a shifted counterpart satisfying φ at any other. Whereas **TM** validates its bimodal interaction principle by the construction of possible worlds itself, this fails in the broader landscape of products of modal logics.[62] We may also restrict

[60] See Lind and Marcus [42] for a comprehensive treatment.

[61] The debate was initiated by Lamport [44] and later given its "final showdown" by Vardi [45]. Emerson and Halpern's CTL* [46] forms a superset of both LTL and CTL by combining path quantifiers with temporal operators freely rather than having either logic subsume the other.

[62] See Kurucz [31] for a general treatment of the products of modal logics. The task relation is also structurally analogous to the program relation in the dynamic logic presented by Harel, Kozen, and Tiuryn [49], though the present framework constructs possible worlds rather than modeling program executions, and its labels carry abelian group structure that dynamic logic does not require.

quantification to intersecting worlds with $\boxdot$, or to worlds with a common past and present relative to the world and time of evaluation by adding $\boxed{\triangleright}$. HyperLTL provides a parallel treatment by extending LTL with quantifiers $\forall\pi$ and $\exists\pi$ ranging over the execution traces of a system. Rather than a bimodal logic with tense and modal operators, HyperLTL is a tense logic with first-order quantifiers for traces, making satisfiability in HyperLTL undecidable.[63] By contrast, the semantics for $\Box$ quantifies over the possible worlds in $H_{\mathcal{F}}$ without binding traces within the object language, making $\mathbf{TM}^+$ decidable as implemented in the Lean 4 repository for this paper.

The *Logic of Tense and Modality* $\mathbf{TM}^+$ provides resources for reasoning about non-deterministic dynamical systems, combining tense and modal operators to give expression to past and future contingency. Incorporating propositional quantifiers would make higher-order quantification over propositions understood as sets of world states available in the object language, bringing the framework closer to the language that Williamson [4, 5] employs. The agentive choice function that Belnap, Perloff, and Xu [41] add to a branching theory of time partitions the intersecting histories at a moment into an agent's available choices. Introducing this overlay would define agentive operators over the dynamics developed here.[64] Probabilistic extensions would connect the framework to stochastic dynamical systems. A categorical treatment of task frames as time-symmetric interval presheaves that are graded by the abelian group of durations with the converse involution definitional offers a further resources. These directions remain to be elaborated elsewhere.

5 Appendix

This section presents the results supporting the main body of the paper. In **§5.1**, I characterize $\Box$ as the strongest objective normal modal operator, proving it unique up to equivalence and deriving the **S4** and **B** axioms, where an axiomatic theory of the objective modalities establishes existence outright in **T2**. Although I will not assume Booleanism, existence is also derived under a coarse-grained strengthening of propositional identity in **C1** to draw a direct comparison with Bacon [55].

After **§5.2** presents results to support the arguments against the two-dimensional semantics in **§2**, the task semantics is developed in **§5.3**, where **§5.4** provides elements of the soundness theorem. The language $\mathcal{L}^+$ extends $\mathcal{L}$ in **§5.5** where **C7** establishes completeness for the corresponding proof systems. The results throughout are formalized in Lean 4 in the repository.[65]

5.1 Objective Modality

This section characterizes metaphysical necessity as the *strongest objective normal modal operator* to derive the **B** and **4** axioms. Objective modality is axiomatized by a predicate O on operator terms together with an auxiliary operator constant B that

[63] Clarkson et al. [50] introduced HyperLTL, and Finkbeiner and Zimmermann [51] showed that it translates into a proper fragment of first-order logic with an equal-level predicate relating positions at the same time on distinct traces. Finkbeiner and Hahn [52] proved that its satisfiability problem is undecidable, whereas Finkbeiner, Rabe, and Sánchez [53] showed that model checking against a given finite-state system is decidable, with complexity rising by an exponential for each quantifier alternation.

[64] See Horty [54] for a `STIT`-based treatment of agentive operators.

[65] See https://github.com/benbrastmckie/BimodalLogic for the Lean 4 repository for this paper.

satisfies this characterization, establishing existence in **T2** before showing that $\Box$ is equivalent to B in **C2**. The treatment is predicative where operator comprehension is confined to formulas containing no operator variables and no occurrences of O. This keeps the system consistent with a fine-grained theory of propositional identity.[66] Since only the modal operator is at issue, I will take $\mathcal{L}^{\Box}$ to be the purely modal fragment of $\mathcal{L}$— that is, $\mathcal{L}$ without $\boxed{\scriptstyle P}$ and $\boxed{\scriptstyle F}$— in which the p_i are henceforth read as *propositional variables* rather than sentence letters. The fragment $\mathcal{L}^{\Box}$ is extended by the propositional identity operator $\equiv$ together with higher-order quantifiers binding variables in both sentence and unary operator propositions.

D1 An *identity extension of* $\mathcal{L}^{\Box}$ is a language $\mathcal{L}^{\equiv}$ that is enriched to include a binary propositional identity operator $\equiv$, read ⌜For φ just is for ψ⌝, whose logic comprises classical propositional logic and the minimal theory of identity given below, where $\chi_{(\psi/\varphi)}$ is the result of replacing one or more occurrences of φ in any formula χ with ψ:

Ref $\vdash \varphi \equiv \varphi$.

Imp $\vdash (\varphi \equiv \psi) \to (\varphi \to \psi)$.

LL $\vdash (\varphi \equiv \psi) \to (\chi \to \chi_{(\psi/\varphi)})$ where ψ is free for φ in χ.[67]

The restriction on substitution is vacuous in $\mathcal{L}^{\equiv}$ while accommodating extensions that include propositional quantifiers. The theory of propositional identity need not be Boolean, accommodating theories in which the absorption laws or other Boolean identities do not hold.[68] Symmetry and transitivity of $\equiv$ are nevertheless derivable.

D2 The *quantified identity extension* $\mathcal{L}^{\equiv}_{2}$ of $\mathcal{L}^{\equiv}$ adds a countable stock of one-place operator variables $\mathcal{Q}, \mathcal{P}, \ldots$, propositional quantifiers $\forall p$ binding the propositional variables, and operator quantifiers $\forall \mathcal{Q}$ binding the operator variables. The *operator terms* are generated from the operator variables, the trivial operator I, the modal primitive $\Box$, and a primitive operator constant B closed under composition $\circ$ and converse $\smile$:[69]

$$T,\ T' \ ::= \ \mathcal{Q} \mid \mathsf{I} \mid \Box \mid \mathsf{B} \mid T \circ T' \mid T^{\smile},$$

and the *well-formed formulas* are defined recursively:

$$\varphi,\ \psi \ ::= \ p_i \mid \bot \mid \varphi \to \psi \mid \varphi \equiv \psi \mid T\varphi \mid \mathsf{O}(T) \mid \forall p\, \varphi \mid \forall \mathcal{Q}\, \varphi.$$

Since each operator term T applies to a single formula, every operator is one-place and the operator quantifier $\forall \mathcal{Q}$ binds one-place operator variables. The formula $\mathsf{O}(T)$ reads ⌜T is an objective modality⌝, where the predicate O and the B are governed by the postulates below. The dual of an operator T is defined as $\langle T \rangle\varphi \coloneqq \neg T \neg \varphi$.

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[66] Predicativity is a standard response to Russell–Myhill. See Walsh [56] for an explicit consistency proof in Church's intensional logic. Predicativity could be dropped by adopting Bacon's [55] *Rule of Equivalence* **D9** to coarsen $\equiv$ enough to block Russell–Myhill on its own as shown in **C1** to make the comparison direct.

[67] A formula ψ is *free for* φ in χ just in case no variable free in ψ or φ is bound at a replaced occurrence in χ, and so satisfied vacuously in $\mathcal{L}^{\equiv}$ which has no propositional quantifiers.

[68] I defend a bilateral theory of propositional identity in Brast-McKie [57].

[69] B must enter as a primitive constant rather than a defined operator term: its defining condition $\forall \mathcal{P}(\mathsf{O}(\mathcal{P}) \to \mathcal{P}p)$ contains both an operator variable and an occurrence of O, each barred from predicative comprehension, so no comprehension instance can express it.

The three operator constants play different theoretical roles. The trivial operator I is fixed outright by **I1** below and serves only as the identity of composition. The constant B is fixed by **O-Meet** to be the meet of the objective modalities; it is an auxiliary, introduced so that a strongest objective modality can be shown to exist without presupposing its identity. The modal primitive $\Box$, by contrast, is governed by no axiom whatever, neither here nor in **D3** below. This silence is deliberate since $\Box$ is the operator under study, and leaving it unconstrained is what makes the hypothesis that it is strongest among the objective modalities— $\mathrm{Str}^{\mathsf{O}}_{L}(\Box)$, in the notation of **D7**— a substantive claim about metaphysical necessity rather than a stipulation about a symbol. Identifying $\Box$ with B outright would instead make **C2** assume what it is meant to establish. The predicate O is likewise primitive rather than defined where it is in terms of O that both the auxiliary constant and the hypothesis about $\Box$ are stated.

D3 A *logic* L *for* $\mathcal{L}_2^{\equiv}$ extends the logic from **D1** to include the axioms and rules below, where $\varphi[\psi/p]$ replaces every free occurrence of p by ψ with ψ free for p, and $\varphi[T/\mathcal{Q}]$ replaces every free occurrence of $\mathcal{Q}$ by the operator term T with T free for $\mathcal{Q}$:

I1 $\vdash \mathsf{I}\varphi \equiv \varphi$

I3 $\vdash (T \circ T')\varphi \equiv T(T'\varphi)$

I2 $\vdash \forall p\, \varphi \to \varphi[\psi/p]$

I4 $\vdash \forall \mathcal{Q}\, \varphi \to \varphi[T/\mathcal{Q}]$

I5 $\vdash \varphi$ implies both $\vdash \forall p\, \varphi$ and $\vdash \forall \mathcal{Q}\, \varphi$

I6 $\vdash \forall p(\varphi \to \psi) \to (\varphi \to \forall p\, \psi)$ provided the variable p is not free in φ

I7 $\vdash \forall \mathcal{Q}(\varphi \to \psi) \to (\varphi \to \forall \mathcal{Q}\, \psi)$ provided the variable $\mathcal{Q}$ is not free in φ

I8 $\vdash \forall \mathcal{Q} \forall p((\mathcal{Q}^{\smile})^{\smile} p \leftrightarrow \mathcal{Q}p)$

I9 $\vdash \forall \mathcal{Q} \forall p(p \to \mathcal{Q}^{\smile}\langle \mathcal{Q}\rangle p)$

I10 $\vdash \forall \mathcal{Q} \forall p(\langle \mathcal{Q}\rangle \mathcal{Q}^{\smile} p \to p)$

I11 $\vdash \exists \mathcal{Q}\, \forall p\, (\mathcal{Q}p \equiv \theta)$ for θ containing no operator variables, no occurrences of O, and at most p free (where $\exists \mathcal{Q}$ abbreviates $\neg \forall \mathcal{Q} \neg$).

D4 Objective modality is characterized with the following operator predicates:

Tra $\mathrm{Tra}(\mathcal{Q}) \coloneqq \forall p \forall q[(p \equiv q) \to (\mathcal{Q}p \equiv \mathcal{Q}q)]$.

Fac $\mathrm{Fac}(\mathcal{Q}) \coloneqq \forall p(\mathcal{Q}p \to p)$.

Obj $\mathrm{Obj}(\mathcal{Q}) \coloneqq \mathrm{Tra}(\mathcal{Q}) \wedge \mathrm{Fac}(\mathcal{Q})$.

Ax $\mathrm{Ax}(\mathcal{Q}) \coloneqq \mathcal{Q}\top \wedge \forall p \forall q[\mathcal{Q}(p \to q) \to (\mathcal{Q}p \to \mathcal{Q}q)]$, where $\top \coloneqq (\bot \to \bot)$.[70]

Ord $\mathcal{Q} \preceq \mathcal{P} \coloneqq \forall p(\mathcal{Q}p \to \mathcal{P}p)$, read ⌜$\mathcal{Q}$ is at least as strong as $\mathcal{P}$⌝.

T1 $\vdash \forall \mathcal{Q}\, \mathrm{Tra}(\mathcal{Q})$.

[70]The conjuncts $\mathcal{Q}\top$ and the distribution schema are the modal axioms **N** and **K**.

Proof. $\vdash \mathcal{Q}p \equiv \mathcal{Q}p$ by **Ref**, and so $\vdash (p \equiv q) \to [(\mathcal{Q}p \equiv \mathcal{Q}p) \to (\mathcal{Q}p \equiv \mathcal{Q}q)]$ follows from **LL** with $\varphi \coloneqq p$, $\psi \coloneqq q$, and $\chi \coloneqq (\mathcal{Q}p \equiv \mathcal{Q}p)$. Permuting antecedents gives $\vdash (\mathcal{Q}p \equiv \mathcal{Q}p) \to [(p \equiv q) \to (\mathcal{Q}p \equiv \mathcal{Q}q)]$, and so $\vdash (p \equiv q) \to (\mathcal{Q}p \equiv \mathcal{Q}q)$ by modus ponens. Generalizing with **I5** derives $\vdash \forall \mathcal{Q}\, \mathrm{Tra}(\mathcal{Q})$ as desired. $\square$

D5 $\mathcal{Q}$ is a *normal modal operator in a logic L*— $\mathrm{Norm}_L(\mathcal{Q})$— iff:

1. $\vdash \mathrm{Ax}(\mathcal{Q})$; and
2. L is closed under $\mathcal{Q}$-necessitation: $\vdash \varphi$ implies $\vdash \mathcal{Q}\varphi$.

D6 A logic L for $\mathcal{L}_2^{\equiv}$ is a *theory of objective modalities* just in case it includes the postulates and rule below:

O-Fac $\vdash \forall \mathcal{Q}(\mathsf{O}(\mathcal{Q}) \to \mathrm{Fac}(\mathcal{Q}))$.

O-Ax $\vdash \forall \mathcal{Q}(\mathsf{O}(\mathcal{Q}) \to \mathrm{Ax}(\mathcal{Q}))$.

O-Nec $\vdash \varphi$ implies $\vdash \forall \mathcal{Q}(\mathsf{O}(\mathcal{Q}) \to \mathcal{Q}\varphi)$.

O-Meet $\vdash \forall p\, (\mathsf{B}p \leftrightarrow \forall \mathcal{P}(\mathsf{O}(\mathcal{P}) \to \mathcal{P}p))$ and $\vdash \mathsf{O}(\mathsf{B})$.

O-Comp $\vdash \forall \mathcal{Q} \forall \mathcal{P}[(\mathsf{O}(\mathcal{Q}) \wedge \mathsf{O}(\mathcal{P})) \to \mathsf{O}(\mathcal{Q} \circ \mathcal{P})]$.[71]

O-Conv $\vdash \forall \mathcal{Q}(\mathsf{O}(\mathcal{Q}) \to \mathsf{O}(\mathcal{Q}^{\smile}))$.

These postulates are jointly consistent as shown in **P2**. Whereas **O-Fac** and **O-Ax** require every objective modality to be factive and to satisfy **N** and **K**, the rule **O-Nec** closes L under necessitation for all of them at once. **O-Meet** asserts that the objective modalities are closed under their infinitary conjunction: B requires just what every objective modality requires, and is itself objective.[72] **O-Comp** closes the objective modalities under composition. Finally, **O-Conv** closes the objective modalities under converse, which is what lifts the theory from **S4** to **S5**.[73]

L1 $\vdash \forall \mathcal{Q}(\mathsf{O}(\mathcal{Q}) \to \mathrm{Obj}(\mathcal{Q}))$ *and any operator term T with* $\vdash \mathsf{O}(T)$ *is normal in L.*

Proof. Given **T1**, **O-Fac** yields $\vdash \forall \mathcal{Q}(\mathsf{O}(\mathcal{Q}) \to \mathrm{Obj}(\mathcal{Q}))$: the primitive O axiomatizes a class whose members satisfy the necessary conditions collected in Obj. For normality, we get $\vdash \mathrm{Ax}(T)$ by instantiating **O-Ax** at T, and L is closed under T-necessitation since **O-Nec** instantiates to $\vdash \mathsf{O}(T) \to T\varphi$. $\square$

D7 $\mathcal{Q}$ is a *strongest objective normal modal operator in L*— $\mathrm{Str}^{\mathsf{O}}_L(\mathcal{Q})$— iff:

[71] Normality is, moreover, preserved under composition: if $\mathrm{Norm}_L(\mathcal{Q})$, then $\vdash \mathrm{Ax}(\mathcal{Q} \circ \mathcal{Q})$— **N** follows by $\mathcal{Q}$-necessitation from $\vdash \mathcal{Q}\top$, and **K** by distributing $\mathcal{Q}$ over an instance of its own distribution schema— while closure under $(\mathcal{Q} \circ \mathcal{Q})$-necessitation is two applications of $\mathcal{Q}$-necessitation.

[72] **O-Nec** and **O-Meet** correspond to *Necessitation* and *L-Necessity* in Bacon and Zeng's theory of necessities [58], where **Theorem 3.1** is the corresponding existence result. **O-Comp** corresponds to their *Composition*, which is a theorem [58, Prop. 2.3] rather than a postulate. The present theory differs by requiring objective modality to be factive **O-Fac**, a strengthening they note but do not adopt [58, fn. 18].

[73] Some such closure postulate is needed rather than merely convenient. Bacon [55, Thm. 5.1] shows that the modal fragment of **HFE** is exactly **S4**, and Bacon and Zeng [58, Cor. 6.5] establish the same bound for their theory of necessities, so the **B** axiom is underivable from the remaining postulates alone.

1. $\vdash \mathsf{O}(\mathcal{Q})$; and
2. $\vdash \forall \mathcal{P}[\mathsf{O}(\mathcal{P}) \to (\mathcal{Q} \preceq \mathcal{P})]$.

Objectivity and normality need not be stated separately: clause (1) already entails $\vdash \mathrm{Obj}(\mathcal{Q})$, $\vdash \mathrm{Ax}(\mathcal{Q})$, and $\mathrm{Norm}_L(\mathcal{Q})$ by **L1**.

T2 $\mathrm{Str}^{\mathsf{O}}_L(\mathsf{B})$, *so L includes a strongest objective normal modal operator.*

Proof. Clause (1) is the second conjunct of **O-Meet**. For clause (2), the first conjunct of **O-Meet** yields $\vdash \mathsf{B}p \to (\mathsf{O}(\mathcal{P}) \to \mathcal{P}p)$ by instantiating at p and $\mathcal{P}$, so permuting antecedents and generalizing gives $\vdash \forall \mathcal{P}[\mathsf{O}(\mathcal{P}) \to (\mathsf{B} \preceq \mathcal{P})]$. By **L1** instantiated at B, using $\vdash \mathsf{O}(\mathsf{B})$ from clause (1), B is objective and normal in L. □

L2 *If* $\mathrm{Str}^{\mathsf{O}}_L(\mathcal{Q})$ *and* $\mathrm{Str}^{\mathsf{O}}_L(\mathcal{P})$, *then* $\vdash \forall p(\mathcal{Q}p \leftrightarrow \mathcal{P}p)$.

Proof. Instantiating the dominance clause for $\mathcal{Q}$ at $\mathcal{P}$ and detaching with $\vdash \mathsf{O}(\mathcal{P})$ yields $\vdash \forall p(\mathcal{Q}p \to \mathcal{P}p)$. Parity of reasoning yields $\vdash \forall p(\mathcal{P}p \to \mathcal{Q}p)$. □

T3 *If* $\mathrm{Str}^{\mathsf{O}}_L(\mathcal{Q})$, *then* $\vdash \forall p(\mathcal{Q}p \to \mathcal{Q}\mathcal{Q}p)$.

Proof. Since $\vdash \mathsf{O}(\mathcal{Q})$, **O-Comp** yields $\vdash \mathsf{O}(\mathcal{Q} \circ \mathcal{Q})$, and so $\vdash \forall p(\mathcal{Q}p \to (\mathcal{Q} \circ \mathcal{Q})p)$ by the dominance clause. The composition axiom $(\mathcal{Q} \circ \mathcal{Q})p \equiv \mathcal{Q}\mathcal{Q}p$ together with **Imp** then yields $\vdash \forall p(\mathcal{Q}p \to \mathcal{Q}\mathcal{Q}p)$. In particular, $\vdash \forall p(\mathsf{B}p \to \mathsf{B}\mathsf{B}p)$ unconditionally, given **T2**, and $\vdash \forall p(\Box p \to \Box\Box p)$ under the hypothesis $\mathrm{Str}^{\mathsf{O}}_L(\Box)$ of **C2**. □

T4 *If* $\mathrm{Str}^{\mathsf{O}}_L(\mathcal{Q})$, *then* $\vdash \forall p(p \to \mathcal{Q}\langle \mathcal{Q} \rangle p)$.[74]

Proof. Since $\vdash \mathsf{O}(\mathcal{Q})$ by clause (1), instantiating **O-Conv** at $\mathcal{Q}$ (**I4**) and detaching yields $\vdash \mathsf{O}(\mathcal{Q}^{\smile})$. Instantiating the dominance clause for $\mathcal{Q}$ at $\mathcal{Q}^{\smile}$ (**I4**) and detaching with $\vdash \mathsf{O}(\mathcal{Q}^{\smile})$ gives $\vdash \forall p(\mathcal{Q}p \to \mathcal{Q}^{\smile}p)$. Instantiating **O-Ax** at $\mathcal{Q}$ and at $\mathcal{Q}^{\smile}$ (**I4**), detaching with $\vdash \mathsf{O}(\mathcal{Q})$ and $\vdash \mathsf{O}(\mathcal{Q}^{\smile})$, yields $\mathrm{Ax}(\mathcal{Q})$ and $\mathrm{Ax}(\mathcal{Q}^{\smile})$. Monotonicity of $\mathcal{Q}$ is thereby a derived rule: given $\vdash \alpha \to \beta$, **O-Nec** together with $\vdash \mathsf{O}(\mathcal{Q})$ yields $\vdash \mathcal{Q}(\alpha \to \beta)$, and the **K** conjunct of $\mathrm{Ax}(\mathcal{Q})$ then yields $\vdash \mathcal{Q}\alpha \to \mathcal{Q}\beta$ by detachment; the same argument using $\vdash \mathsf{O}(\mathcal{Q}^{\smile})$ and $\mathrm{Ax}(\mathcal{Q}^{\smile})$ yields monotonicity for $\mathcal{Q}^{\smile}$. Contraposing, applying $\mathcal{Q}$-monotonicity or $\mathcal{Q}^{\smile}$-monotonicity, and contraposing again yields monotonicity of $\langle \mathcal{Q} \rangle$ and of $\langle \mathcal{Q}^{\smile} \rangle$: given $\vdash \alpha \to \beta$, $\vdash \langle \mathcal{Q} \rangle \alpha \to \langle \mathcal{Q} \rangle \beta$, and likewise for $\langle \mathcal{Q}^{\smile} \rangle$. Instantiating $\vdash \forall p(\mathcal{Q}p \to \mathcal{Q}^{\smile}p)$ at $\neg p$ (**I2**) and contraposing yields $\vdash \langle \mathcal{Q}^{\smile} \rangle p \to \langle \mathcal{Q} \rangle p$. Instantiating this at $p \coloneqq \mathcal{Q}^{\smile}q$ (**I2**) gives $\vdash \langle \mathcal{Q}^{\smile} \rangle \mathcal{Q}^{\smile}q \to \langle \mathcal{Q} \rangle \mathcal{Q}^{\smile}q$, which chains by hypothetical syllogism with **I10** instantiated at $\mathcal{Q}$—$\vdash \langle \mathcal{Q} \rangle \mathcal{Q}^{\smile}q \to q$—to give $\vdash \langle \mathcal{Q}^{\smile} \rangle \mathcal{Q}^{\smile}q \to q$. Instantiating **I9** at $\mathcal{Q}^{\smile}$ (**I4**) gives $\vdash \forall p(p \to (\mathcal{Q}^{\smile})^{\smile}\langle \mathcal{Q}^{\smile} \rangle p)$, and instantiating **I8** at $\mathcal{Q}$ and at $\langle \mathcal{Q}^{\smile} \rangle p$ gives $\vdash (\mathcal{Q}^{\smile})^{\smile}\langle \mathcal{Q}^{\smile} \rangle p \leftrightarrow \mathcal{Q}\langle \mathcal{Q}^{\smile} \rangle p$, which

[74] **O-Conv** is a term-level form of the principle, due to Williamson [4, p. 457], that every necessity has a reversal, where Bacon and Zeng [58, Prop. 4.1] prove that principle equivalent to the **B** axiom for the broadest necessity. Here **I9** makes $\mathcal{Q}^{\smile}$ a reversal of $\mathcal{Q}$ and **O-Conv** keeps it in the class, so **T4** derives one direction of that equivalence with a canonical converse in place of an existential.

chains with the left-to-right conditional of this biconditional by hypothetical syllogism to give $\vdash \forall p(p \to \mathcal{Q}\langle\mathcal{Q}^{\smile}\rangle p)$. Applying $\mathcal{Q}$-monotonicity to $\vdash \langle\mathcal{Q}^{\smile}\rangle\mathcal{Q}^{\smile}q \to q$ yields $\vdash \mathcal{Q}\langle\mathcal{Q}^{\smile}\rangle\mathcal{Q}^{\smile}q \to \mathcal{Q}q$, which chains with $\vdash \forall p(p \to \mathcal{Q}\langle\mathcal{Q}^{\smile}\rangle p)$ instantiated at $\mathcal{Q}^{\smile}q$ (**I2**)— $\vdash \mathcal{Q}^{\smile}q \to \mathcal{Q}\langle\mathcal{Q}^{\smile}\rangle\mathcal{Q}^{\smile}q$— by hypothetical syllogism to give $\vdash \mathcal{Q}^{\smile}q \to \mathcal{Q}q$. Instantiating **I9** at $\mathcal{Q}$ (**I4**) gives $\vdash \forall p(p \to \mathcal{Q}^{\smile}\langle\mathcal{Q}\rangle p)$. Instantiating $\vdash \mathcal{Q}^{\smile}q \to \mathcal{Q}q$ at $q \coloneqq \langle\mathcal{Q}\rangle p$ (**I2**) gives $\vdash \mathcal{Q}^{\smile}\langle\mathcal{Q}\rangle p \to \mathcal{Q}\langle\mathcal{Q}\rangle p$, which chains with $\vdash p \to \mathcal{Q}^{\smile}\langle\mathcal{Q}\rangle p$ by hypothetical syllogism, and generalizing on p (**I5**) gives $\vdash \forall p(p \to \mathcal{Q}\langle\mathcal{Q}\rangle p)$. In particular, $\vdash \forall p(p \to \mathsf{B}\langle\mathsf{B}\rangle p)$ unconditionally, given **T2**, and $\vdash \forall p(p \to \Box\langle\Box\rangle p)$ under the hypothesis $\mathrm{Str}^{\mathsf{O}}_{L}(\Box)$ of **C2**. □

With existence settled by **T2**, it remains substantive which operators fall under O and so coincide with B.

D8 $\boxed{\top}\varphi \coloneqq (\varphi \equiv \top)$. Applying comprehension (**I11**) to the formula $p \equiv \top$— which contains no operator variables and no occurrences of O— guarantees an operator satisfying $\forall p(\mathcal{Q}p \equiv \boxed{\top}p)$.

P1 *The operator* $\boxed{\top}$ *of* ***D8*** *is objective,* $\vdash \boxed{\top}\top$*, and* $\vdash \forall\mathcal{P}[\mathrm{Ax}(\mathcal{P}) \to (\boxed{\top} \preceq \mathcal{P})]$*; hence* $\boxed{\top} \preceq \mathsf{B}$.[75]

Proof. Transparency is an instance of **T1**, and factivity holds since $p \equiv \top$ yields $\top \equiv p$ by symmetry, whence p by **Imp**, so $\boxed{\top}$ is objective. Moreover, $\vdash \boxed{\top}\top$ by **Ref**, and $\vdash \forall\mathcal{P}[\mathrm{Ax}(\mathcal{P}) \to (\boxed{\top} \preceq \mathcal{P})]$ since $\mathrm{Ax}(\mathcal{P})$ yields $\mathcal{P}\top$ and **LL** carries $\mathcal{P}\top$ to $\mathcal{P}p$ given $\top \equiv p$. In particular $\boxed{\top} \preceq \mathsf{B}$, since $\vdash \mathrm{Ax}(\mathsf{B})$ by **T2**. □

D9 A logic L for $\mathcal{L}_2^{\equiv}$ includes the *Rule of Equivalence* just in case $\vdash \varphi \leftrightarrow \psi$ implies $\vdash \varphi \equiv \psi$, so that interderivable formulas are identical (Bacon [55, p. 746]).

C1 *If L includes the Rule of Equivalence* ***D9****, then L includes a strongest objective normal modal operator in the unrelativized sense— objective, normal, and $\preceq$-below every operator satisfying* Ax*— unique up to equivalence and given by* $\boxed{\top}$.[76]

Proof. By **P1**, only normality remains, which the *Rule of Equivalence* supplies. For necessitation, $\vdash \varphi$ makes φ and $\top$ interderivable, so $\vdash \varphi \equiv \top$, i.e., $\vdash \boxed{\top}\varphi$. For distribution, **LL** carries $(p \to q) \equiv \top$ to $(\top \to q) \equiv \top$ given $p \equiv \top$, which the rule's instance $(\top \to q) \equiv q$ collapses to $q \equiv \top$. Uniqueness holds since any two normal operators

[75] Yet the postulates place no particular operator besides B in O, so $\vdash \mathsf{O}(\boxed{\top})$ is not derivable, and in fine-grained settings it is precisely the normality of $\boxed{\top}$ that fails: existence no longer rests on this candidate; see **5.1**. See Bacon [55, §3 and fn. 28] for related discussion.

[76] Bacon [55, p. 747] shows that **D9** is stronger than *Booleanism*— that Boolean-equivalent formulas are identical— which yields distribution but not necessitation. This corollary transposes Bacon's results into the present setting for comparison: $\boxed{\top}$ is the Cresswell–Suszko operator, for which Bacon proves **S4** (Prop. 3.1), that it is the broadest necessity (Prop. 4.1), and that it is unique given a further functionality axiom (Prop. 4.2), with propositional identity reducing to necessary equivalence (Prop. 3.2). The present statement differs in characterizing $\Box$ by objectivity rather than broadness alone and in assuming only predicative comprehension over a minimal theory of $\equiv$. Given the Rule of Equivalence, propositional identity is coarse-grained enough to block Russell–Myhill on its own, and so the predicative restriction on comprehension could be dropped. However, neither the rule nor Booleanism is otherwise assumed.

dominant in this sense satisfy Ax, so instantiating each dominance clause at the other yields equivalence. $\square$

Remark 5.1 Unconditionally, $\boxed{\top} \preceq \mathsf{B}$ (**P1**); under the Rule of Equivalence, $\boxed{\top}$ is strongest in the unrelativized sense by **C1**, though the postulates still do not deliver $\vdash \mathsf{O}(\boxed{\top})$. If $\mathsf{O}(\boxed{\top})$ is additionally assumed— as the intended interpretation validates, $\boxed{\top}$ being materially the unrestricted quantifier under the rule— then $\boxed{\top}$ satisfies clause (1) of **D7** directly and clause (2) via **O-Ax**, so **L2** yields $\vdash \forall p(\mathsf{B}p \leftrightarrow \boxed{\top}p)$.

P2 *The theory of objective modalities **D6** is consistent.*

Proof. Interpret operator variables as ranging over all operations on propositions, and interpret O as true of exactly the restricted world-quantifiers $\mathcal{Q}_R$ for reflexive relations $R \subseteq H_{\mathcal{F}} \times H_{\mathcal{F}}$, where $\mathcal{Q}_R\varphi$ is true at (τ, x) just in case φ is true at (σ, x) for every σ with $\langle \tau, \sigma \rangle \in R$. Reflexivity validates **O-Fac**; each $\mathcal{Q}_R$ is a universal quantifier, validating **O-Ax** and **O-Nec**; and the composition of reflexive relations is reflexive, validating **O-Comp**. The converse of a reflexive relation is again reflexive, validating **O-Conv**. Interpreting $\mathcal{Q}_R^{\smile}$ as $\mathcal{Q}_{R^{\smile}}$, where $R^{\smile} \coloneqq \{\langle \sigma, \tau \rangle \mid \langle \tau, \sigma \rangle \in R\}$, validates both **I9** and **I10** for *arbitrary* R, reflexivity not required: given $p(\tau)$, $\mathcal{Q}_{R^{\smile}}\langle \mathcal{Q}_R \rangle p$ at τ reduces to $\forall \sigma(R\sigma\tau \to \exists \rho(R\sigma\rho \wedge p(\rho)))$, witnessed by $\rho \coloneqq \tau$, giving the unit; and $\langle \mathcal{Q}_R \rangle \mathcal{Q}_{R^{\smile}} q$ at τ gives some σ with $R\tau\sigma$ and $\forall \rho(R\rho\sigma \to q(\rho))$, whence $q(\tau)$ by instantiating at $\rho \coloneqq \tau$, giving the counit. Involution holds since $(R^{\smile})^{\smile} = R$ for any relation. The family of reflexive relations was already closed under converse, so the union is still the total relation and B's denotation as the unrestricted $\square$ is unchanged: the seam being closed is purely derivational, and **O-Conv** changes no denotation. For **O-Meet**, the condition $\forall \mathcal{P}(\mathsf{O}(\mathcal{P}) \to \mathcal{P}p)$ quantifies over the worlds accessible by some reflexive relation, and the union of all reflexive relations is the total relation $H_{\mathcal{F}} \times H_{\mathcal{F}}$— itself reflexive— whose quantifier is the unrestricted $\square$. Interpreting B as $\square$ thus validates both conjuncts: the least objective modality is the unrestricted quantifier, as the paper maintains. Comprehension (**I11**) holds since any O-free θ defines an operation on propositions, and the model in fact validates arbitrary meets of objective modalities, more than **O-Meet** asserts. $\square$

C2 *If* $\mathrm{Str}_L^{\mathsf{O}}(\square)$, *then* $\vdash \forall p(\square p \leftrightarrow \mathsf{B}p)$, $\square$ *has the **S4** axiom and the **B** axiom, and* $\square$ *is factive and closed under necessitation— together,* $\square$ *has an **S5** logic.*

Proof. Existence is categorical: **T2** delivers the strongest objective normal modal operator B outright. Given $\mathrm{Str}_L^{\mathsf{O}}(\square)$— equivalently, $\vdash \mathsf{O}(\square)$ together with the dominance clause— **L2** yields $\vdash \forall p(\square p \leftrightarrow \mathsf{B}p)$, **T3** yields the **S4** axiom, **T4** yields the **B** axiom, and factivity and necessitation for the primitive $\square$ follow by detaching **O-Fac** and **O-Nec** at $\square$, together delivering an **S5** logic for $\square$. $\square$

5.2 Two-Dimensional Semantics

This section concerns the two-dimensional models introduced in §2.2. Definitions will be restated for convenience throughout.

D10 A *two-dimensional model* is a structure $\mathcal{M} = \langle W, T, \leqslant, |\cdot| \rangle$ where:

Worlds: A nonempty set of *worlds* W.

Times: A nonempty set of *times* T.

Order: A weak total order $\leqslant$ on T.

Interpretation: An interpretation function $|\cdot|$ assigning a set of world-time pairs $|p_i| \subseteq W \times T$ to each $p_i \in \mathbb{L}$.

D11 The language $\mathcal{L}^{\mathrm{M}} \coloneqq \langle \mathbb{L}, \bot, \rightarrow, \boxtimes, \boxed{\scriptstyle P}, \boxed{\scriptstyle F} \rangle$ where $\mathbb{L} \coloneqq \{p_i : i \in \mathbb{N}\}$ is a countable set of sentence letters where the remaining symbols denote falsity, material implication, the disputed modal operator that Thomason reads 'necessarily always', the universal past tense operator, and the universal future tense operator, respectively.

D12 Truth in a two-dimensional model at a world and time is defined recursively:

(p_i) $\mathcal{M}, w, x \vDash p_i$ *iff* $\langle w, x \rangle \in |p_i|$.

($\bot$) $\mathcal{M}, w, x \nvDash \bot$.

($\rightarrow$) $\mathcal{M}, w, x \vDash \varphi \rightarrow \psi$ *iff* $\mathcal{M}, w, x \nvDash \varphi$ or $\mathcal{M}, w, x \vDash \psi$.

($\boxtimes$) $\mathcal{M}, w, x \vDash \boxtimes\varphi$ *iff* $\mathcal{M}, u, y \vDash \varphi$ for all $u \in W$ and $y \in T$.

($\boxed{\scriptstyle P}$) $\mathcal{M}, w, x \vDash \boxed{\scriptstyle P}\varphi$ *iff* $\mathcal{M}, w, y \vDash \varphi$ for all $y \in T$ where $y < x$.

($\boxed{\scriptstyle F}$) $\mathcal{M}, w, x \vDash \boxed{\scriptstyle F}\varphi$ *iff* $\mathcal{M}, w, y \vDash \varphi$ for all $y \in T$ where $x < y$.

D13 A nonempty relation $Z \subseteq (W_1 \times T_1) \times (W_2 \times T_2)$— writing $(w, x) \rightarrow (v, y)$ to represent pairs in Z— is a $\mathcal{L}^{\mathrm{M}}$-*bisimulation* between $\mathcal{M}_1$ and $\mathcal{M}_2$ just in case whenever $(w, x) \rightarrow (w', x')$, all of the following conditions hold:

Atomic harmony: For all $p_i \in \mathbb{L}$, $\mathcal{M}_1, w, x \vDash p_i$ just in case $\mathcal{M}_2, w', x' \vDash p_i$.

Past Forth: For every y with $y < x$ there is y' with $y' < x'$ and $(w, y) \rightarrow (w', y')$.

Past Back: For every y' with $y' < x'$ there is y with $y < x$ and $(w, y) \rightarrow (w', y')$.

Future Forth: For every y with $x < y$ there is y' with $x' < y'$ and $(w, y) \rightarrow (w', y')$.

Future Back: For every y' with $x' < y'$ there is y with $x < y$ and $(w, y) \rightarrow (w', y')$.

Global Forth: For every $(u, z) \in W_1 \times T_1$ there is $(u', z') \in W_2 \times T_2$ with $(u, z) \rightarrow (u', z')$.

Global Back: For every $(u', z') \in W_2 \times T_2$ there is $(u, z) \in W_1 \times T_1$ with $(u, z) \rightarrow (u', z')$.

L3 *If Z is a $\mathcal{L}^{\mathrm{M}}$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ and $(w, x) \rightarrow (w', x')$, then for every $\mathcal{L}^{\mathrm{M}}$-formula φ, $\mathcal{M}_1, w, x \vDash \varphi$ just in case $\mathcal{M}_2, w', x' \vDash \varphi$.*

Proof. The proof proceeds by induction on the complexity of φ where the case for the sentence letters and extensional operators are routine.

Case $\boxed{\text{F}}$*:* Assume $\varphi = \boxed{\text{F}}\psi$. Supposing for contraposition that $\mathcal{M}_2, w', x' \nvDash \boxed{\text{F}}\psi$, it follows that $\mathcal{M}_2, w', y' \nvDash \psi$ for some $x' < y'$. By *Future Back*, there is a y with $x < y$ where $(w, y) \to (w', y')$, and so $\mathcal{M}_1, w, y \nvDash \psi$ by hypothesis. Thus $\mathcal{M}_1, w, x \nvDash \boxed{\text{F}}\psi$. Contraposition and parity of reasoning establish that $\mathcal{M}_1, w, x \vDash \boxed{\text{F}}\psi$ just in case $\mathcal{M}_2, w', x' \vDash \boxed{\text{F}}\psi$. The other tense operators are similar.

Case $\boxtimes$*:* Assume $\varphi = \boxtimes\psi$. Supposing for contraposition that $\mathcal{M}_2, w', x' \nvDash \boxtimes\psi$, it follows that $\mathcal{M}_2, u', z' \nvDash \psi$ for some $u' \in W_2$ and $z' \in T_2$. By *Global Back*, there is $(u, z) \in W_1 \times T_1$ with $(u, z) \to (u', z')$, and so $\mathcal{M}_1, u, z \nvDash \psi$ by hypothesis. Thus $\mathcal{M}_1, w, x \nvDash \boxtimes\psi$, where contraposition and parity of reasoning complete the case.

By induction on complexity, we may conclude that $\mathcal{M}_1, w, x \vDash \varphi$ just in case $\mathcal{M}_2, w', x' \vDash \varphi$ for all well-formed sentences φ in $\mathcal{L}^{\text{M}}$. $\square$

T5 $\boxed{2}$ *is not definable in* $\mathcal{L}^{\text{M}}$.

Proof. Assume for contradiction that $\boxed{2}$ is definable in $\mathcal{L}^{\text{M}}$ so that $\boxed{2}\varphi$ abbreviates a well-formed sentence of $\mathcal{L}^{\text{M}}$ with the following derived semantic clause:

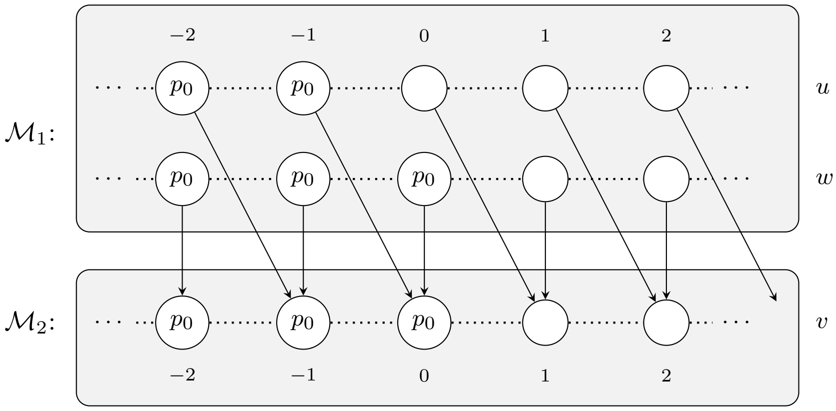
($\boxed{2}$) $\mathcal{M}, w, x \vDash \boxed{2}\varphi$ *iff* $\mathcal{M}, u, x \vDash \varphi$ for all $u \in W$.

We define the models $\mathcal{M}_1 = \langle W_1, T_1, \leqslant, |\cdot|_1\rangle$ and $\mathcal{M}_2 = \langle W_2, T_2, \leqslant, |\cdot|_2\rangle$ where:

$\mathcal{M}_1$: $W_1 = \{w, u\}$, $T_1 = \mathbb{Z}$, and $|p_0|_1 = \{(w, x) : x \leqslant 0\} \cup \{(u, x) : x \leqslant -1\}$, so that w satisfies p_0 at every time up to and including 0, while u satisfies p_0 at every time up to and including -1.

$\mathcal{M}_2$: $W_2 = \{v\}$, $T_2 = \mathbb{Z}$, and $|p_0|_2 = \{(v, x) : x \leqslant 0\}$, so that v satisfies p_0 at every time up to and including 0.

We may then define the relation $Z \subseteq (W_1 \times T_1) \times (W_2 \times T_2)$, writing $(w, x) \to (v, y)$ to represent pairs in Z as before, by $(w, x) \to (v, x)$ for every $x \in \mathbb{Z}$ and $(u, x) \to (v, x + 1)$ for every $x \in \mathbb{Z}$. The following diagram depicts a representative window of the construction, where the ellipses indicate that the pattern continues without bound in both directions:



We verify that Z is a $\mathcal{L}^{\mathrm{M}}$-bisimulation by checking the clauses of D13 for an arbitrary pair in Z as given below:

Atomic harmony: For $(w,x) \rightarrow (v,x)$: $\mathcal{M}_1, w, x \vDash p_0$ just in case $x \leqslant 0$, and $\mathcal{M}_2, v, x \vDash p_0$ just in case $x \leqslant 0$. For $(u,x) \rightarrow (v, x+1)$: $\mathcal{M}_1, u, x \vDash p_0$ just in case $x \leqslant -1$, and $\mathcal{M}_2, v, x+1 \vDash p_0$ just in case $x+1 \leqslant 0$, i.e. $x \leqslant -1$. Both pairs agree on p_0 in every case.

Past Forth and Past Back: For $(w,x) \rightarrow (v,x)$, every $z < x$ satisfies $(w,z) \rightarrow (v,z)$ with $z < x$, giving both directions by taking $z' = z$. For $(u,x) \rightarrow (v,x+1)$, every $z < x$ satisfies $(u,z) \rightarrow (v, z+1)$ with $z+1 < x+1$, giving Past Forth; conversely every $z' < x+1$ in T_2 satisfies $z' - 1 < x$ with $(u, z'-1) \rightarrow (v, z')$, giving Past Back.

Future Forth and Future Back: Symmetric to the previous case: for $(w,x) \rightarrow (v,x)$, take $z' = z$ for every $z > x$ and conversely; for $(u,x) \rightarrow (v,x+1)$, every $z > x$ satisfies $(u,z) \rightarrow (v,z+1)$ with $z+1 > x+1$, and every $z' > x+1$ satisfies $z'-1 > x$ with $(u, z'-1) \rightarrow (v, z')$.

Global Forth: Every $(w,x) \in W_1 \times T_1$ appears via $(w,x) \rightarrow (v,x)$, and every (u,x) appears via $(u,x) \rightarrow (v,x+1)$.

Global Back: Every $(v,x) \in W_2 \times T_2$ is the right-projection of $(w,x) \rightarrow (v,x)$.

Thus Z is a $\mathcal{L}^{\mathrm{M}}$-bisimulation. Given that $(w,0) \rightarrow (v,0)$, it follows by **L3** that for all well-formed φ of $\mathcal{L}^{\mathrm{M}}$, the following biconditional holds:

$$\mathcal{M}_1, w, 0 \vDash \varphi \text{ just in case } \mathcal{M}_2, v, 0 \vDash \varphi. \qquad (*)$$

On the assumption that $\boxed{2}$ is definable in $\mathcal{L}^{\mathrm{M}}$, we may conclude that $\mathcal{M}_1, w, 0 \vDash \boxed{2}p_0$ just in case $\mathcal{M}_2, v, 0 \vDash \boxed{2}p_0$. By the claimed semantic clause for $\boxed{2}$ we have:

$$\begin{aligned} \mathcal{M}_1, w, 0 \vDash \boxed{2}p_0 &\Leftrightarrow \mathcal{M}_1, w, 0 \vDash p_0 \text{ and } \mathcal{M}_1, u, 0 \vDash p_0 \\ \mathcal{M}_2, v, 0 \vDash \boxed{2}p_0 &\Leftrightarrow \mathcal{M}_2, v, 0 \vDash p_0 \end{aligned}$$

Since $\mathcal{M}_1, u, 0 \nvDash p_0$, it follows that $\mathcal{M}_1, w, 0 \nvDash \boxed{\scriptstyle 2}p_0$. Given that v is the only world in W_2 and $\mathcal{M}_2, v, 0 \vDash p_0$, it follows that $\mathcal{M}_2, v, 0 \vDash \boxed{\scriptstyle 2}p_0$. Hence $\mathcal{M}_1, w, 0 \nvDash \boxed{\scriptstyle 2}p_0$ while $\mathcal{M}_2, v, 0 \vDash \boxed{\scriptstyle 2}p_0$, contradicting $(*)$. Thus we may conclude by *reductio* that the operator $\boxed{\scriptstyle 2}$ with the derived semantic clause given above is not definable in $\mathcal{L}^{\mathrm{M}}$. $\square$

D14 Dorr and Goodman [29] extend their language over product models by adding a countable set of time variables $\mathcal{V} \coloneqq \{t_i : i \in \mathbb{N}\}$ which may be bound by first-order quantifiers; transposing this extension to $\mathcal{L}^{\mathrm{M}}$ for comparison yields the language $\mathcal{L}^{\mathrm{F}}$. An *assignment* is a function $g : \mathcal{V} \to T$ from time variables in $\mathcal{V}$ to times in T which is used to extend the semantics. Truth in a two-dimensional model at a world and time relative to an assignment extends the semantics for $\mathcal{L}^{\mathrm{M}}$ to include the following:

($\exists t$) $\mathcal{M}, w, x, g \vDash \exists t\, \varphi$ *iff* $\mathcal{M}, w, x, g' \vDash \varphi$ for some g' differing from g at most in t.

(Present) $\mathcal{M}, w, x, g \vDash \mathrm{Present}(t)$ *iff* $g(t) = x$.

L4 *In $\mathcal{L}^{\mathrm{F}}$, $\boxed{\scriptstyle 2}$ may be defined by $\boxed{\scriptstyle 2}\varphi \coloneqq \exists t\, [\mathit{Present}(t) \wedge \boxtimes(\mathit{Present}(t) \to \varphi)]$.*

Proof. To establish that $\boxed{\scriptstyle 2}$ is definable, we show:

$$\begin{aligned}
&\mathcal{M}, w, x, g \vDash \exists t\, [\mathrm{Present}(t) \wedge \boxtimes(\mathrm{Present}(t) \to \varphi)] \\
&\quad \Leftrightarrow \text{ there is } g' \text{ with } g'(t) = x \text{ and } \mathcal{M}, u, y, g' \vDash \mathrm{Present}(t) \to \varphi \text{ for all } u \in W, y \in T \\
&\quad \Leftrightarrow \text{ there is } g' \text{ with } g'(t) = x \text{ and } \mathcal{M}, u, x, g' \vDash \varphi \text{ for all } u \in W \\
&\quad \Leftrightarrow \mathcal{M}, u, x, g \vDash \varphi \text{ for all } u \in W
\end{aligned}$$

The first equivalence follows from the semantics for $\exists$, $\wedge$, and $\boxtimes$. For the second equivalence, since $g'(t) = x$, we have $\mathcal{M}, u, y, g' \vDash \mathrm{Present}(t) \to \varphi$ just in case either $y \neq x$ or $\mathcal{M}, u, y, g' \vDash \varphi$ with $y = x$. It follows that $\mathcal{M}, u, y, g' \vDash \mathrm{Present}(t) \to \varphi$ for all $u \in W$ and $y \in T$ just in case $\mathcal{M}, u, x, g' \vDash \varphi$ for all $u \in W$. The final equivalence holds because t does not occur free in φ, so g and g' agree on the truth of φ, and so we may choose any t-variant g' of g with $g'(t) = x$ to witness the existential. $\square$

T6 *Given the two-dimensional semantics for $\mathcal{L}^{\mathrm{F}}$, both **P1** and **P2** are invalid.*

Proof. Let $\mathcal{M} = \langle W, T, \leqslant, |\cdot| \rangle$ be a two-dimensional model with worlds $W = \{w, u\}$, times $T = \{0, 1\}$ where $\leqslant$ is the usual order on $\{0, 1\}$, and $|p_0| = \{(w, 0), (u, 0)\}$. Since $(w, 0) \in |p_0|$ and $(u, 0) \in |p_0|$, we have $\mathcal{M}, w, 0, g \vDash p_0$ and $\mathcal{M}, u, 0, g \vDash p_0$ where g is any assignment. It follows by **L4** that $\mathcal{M}, w, 0, g \vDash \boxed{\scriptstyle 2}p_0$.

However, $\mathcal{M}, w, 0, g \vDash \triangle p_0$ just in case $\mathcal{M}, w, y, g \vDash p_0$ for all $y \in T$ with $0 \leqslant y$. Since $0 \leqslant 1$ and $(w, 1) \notin |p_0|$, we have $\mathcal{M}, w, 1, g \nvDash p_0$. Therefore $\mathcal{M}, w, 0, g \nvDash \triangle p_0$. It follows that $\mathcal{M}, w, 0, g \nvDash \boxed{\scriptstyle 2}p_0 \to \triangle p_0$, and so **P1** is invalid. Since **P2** is equivalent to **P1**, it follows that **P2** is also invalid. $\square$

D15 An *order automorphism* on the structure $\langle T, \leqslant \rangle$ is a bijective function $\bar{a} : T \to T$ such that for all $x, y \in T$, we have $x \leqslant y$ just in case $\bar{a}(x) \leqslant \bar{a}(y)$.

D16 Given a two-dimensional model $\mathcal{M} = \langle W, T, \leqslant, |\cdot|\rangle$, the worlds $w, w' \in W$ are *time-shifted from x to y*— written $w \approx_x^y w'$— *iff* there exists an order automorphism $\bar{a} : T \to T$ where $y = \bar{a}(x)$ and for all sentence letters $p_i \in \mathbb{L}$ and times $z \in T$, we have $\langle w, z\rangle \in |p_i|$ just in case $\langle w', \bar{a}(z)\rangle \in |p_i|$.

L5 *If $w \approx_x^y u$ for some $x, y \in T$, then there is some order automorphism $\bar{a} : T \to T$ where $w \approx_z^{\bar{a}(z)} u$ for all $z \in T$.*

Proof. Suppose $w \approx_x^y u$ for $x, y \in T$. By **D16**, $y = \bar{a}(x)$ for some order automorphism $\bar{a} : T \to T$ where the following equivalence holds for all $p_i \in \mathbb{L}$ and $z \in T$:

$$\langle w, z\rangle \in |p_i| \text{ just in case } \langle u, \bar{a}(z)\rangle \in |p_i|. \tag{$*$}$$

Let $z \in T$ be arbitrary. To show $w \approx_z^{\bar{a}(z)} u$, we must exhibit an order automorphism $\bar{b} : T \to T$ satisfying two conditions: (i) $\bar{a}(z) = \bar{b}(z)$, and (ii) for all $p_i \in \mathbb{L}$ and $t \in T$, $\langle w, t\rangle \in |p_i|$ just in case $\langle u, \bar{b}(t)\rangle \in |p_i|$. Taking $\bar{b} = \bar{a}$ satisfies both conditions: (i) holds trivially, and (ii) is exactly ($*$). Since z was arbitrary, $w \approx_z^{\bar{a}(z)} u$ for all $z \in T$. □

D17 A two-dimensional model $\mathcal{M} = \langle W, T, \leqslant, |\cdot|\rangle$ of $\mathcal{L}^{\mathrm{K}}$ is *abundant* just in case for every $w \in W$ and times $x, y \in T$, there is some world $w' \in W$ where $w \approx_x^y w'$.

L6 *$\mathcal{M}_2, w, x \vDash \varphi$ just in case $\mathcal{M}_2, u, y \vDash \varphi$ for any well-formed sentence φ of $\mathcal{L}^{\mathrm{K}}$ and abundant two-dimensional model $\mathcal{M}_2 = \langle W, T, \leqslant, |\cdot|\rangle$ where $w \approx_x^y u$.*

Proof. Assume $w \approx_x^y u$ in $\mathcal{M}_2 = \langle W, T, \leqslant, |\cdot|\rangle$. The proof proceeds by induction on the complexity of φ. By **D16**, there exists an order automorphism $\bar{a} : T \to T$ where $y = \bar{a}(x)$ and for all sentence letters $p_i \in \mathbb{L}$ and times $z \in T$:

$$\langle w, z\rangle \in |p_i| \;\Leftrightarrow\; \langle u, \bar{a}(z)\rangle \in |p_i|. \tag{†}$$

Base Case ($\varphi = p_i$): By (†) with $z = x$, it follows that $\langle w, x\rangle \in |p_i|$ just in case $\langle u, \bar{a}(x)\rangle \in |p_i|$. Since $y = \bar{a}(x)$, this gives $\langle w, x\rangle \in |p_i|$ just in case $\langle u, y\rangle \in |p_i|$. By the semantic clause for sentence letters, $\mathcal{M}_2, w, x \vDash p_i$ just in case $\mathcal{M}_2, u, y \vDash p_i$.

Base Case ($\varphi = \bot$): By the semantic clause for $\bot$, we have $\mathcal{M}_2, w, x \nvDash \bot$ and $\mathcal{M}_2, u, y \nvDash \bot$. Therefore the biconditional holds immediately.

Inductive Case ($\varphi = \psi \to \chi$):

$$\begin{aligned}
\mathcal{M}_2, w, x \vDash \psi \to \chi \;&\Leftrightarrow\; \mathcal{M}_2, w, x \nvDash \psi \text{ or } \mathcal{M}_2, w, x \vDash \chi \\
&\Leftrightarrow\; \mathcal{M}_2, u, y \nvDash \psi \text{ or } \mathcal{M}_2, u, y \vDash \chi \\
&\Leftrightarrow\; \mathcal{M}_2, u, y \vDash \psi \to \chi
\end{aligned}$$

The induction hypothesis justifies the second equivalence when applied to ψ and χ.

Inductive Case ($\varphi = \Box\psi$):

$$\begin{aligned}\mathcal{M}_2, w, x \not\models \Box\psi &\Leftrightarrow \mathcal{M}_2, v, x \not\models \psi \text{ for some } v \in W\\ &\Leftrightarrow \mathcal{M}_2, v', y \not\models \psi \text{ for some } v \in W\\ &\Leftrightarrow \mathcal{M}_2, u, y \not\models \Box\psi\end{aligned}$$

Since $\mathcal{M}_2$ is abundant where $v \in W$ and $x, y \in T$, there is some $v' \in W$ where $v \approx^y_x v'$. It follows by the induction hypothesis that $\mathcal{M}_2, v', y \not\models \psi$, where parity of reasoning establishes the converse. This justifies the second biconditional.

Inductive Case ($\varphi = \boxed{\mathrm{P}}\psi$):

$$\begin{aligned}\mathcal{M}_2, w, x \not\models \boxed{\mathrm{P}}\psi &\Leftrightarrow \mathcal{M}_2, w, z \not\models \psi \text{ for some } z < x\\ &\Leftrightarrow \mathcal{M}_2, u, z' \not\models \psi \text{ for some } z' < y\\ &\Leftrightarrow \mathcal{M}_2, u, y \not\models \boxed{\mathrm{P}}\psi\end{aligned}$$

For the forward direction of the second biconditional, assume that $\mathcal{M}_2, w, z \not\models \psi$ for some $z < x$. Since $\bar{a}$ is an order automorphism, $\bar{a}(z) < \bar{a}(x)$ where $\bar{a}(x) = y$. Letting $z' = \bar{a}(z)$, we have $z' < y$. Since $w \approx^y_x u$ where $y = \bar{a}(x)$ and $z' = \bar{a}(z)$, it follows by **L5** that $w \approx^{z'}_z u$. Applying the induction hypothesis to ψ, we have $\mathcal{M}_2, w, z \models \psi$ just in case $\mathcal{M}_2, u, z' \models \psi$, and so $\mathcal{M}_2, u, z' \not\models \psi$ for some $z' < y$.

For the backward direction of the second biconditional, assume $\mathcal{M}_2, u, z' \not\models \psi$ for some $z' < y$. Since $\bar{a} : T \to T$ is an order automorphism, we have $\bar{a}^{-1}(z') < \bar{a}^{-1}(y)$. Letting $z = \bar{a}^{-1}(z')$, we have $z < x$ given that $x = \bar{a}^{-1}(y)$. Since $w \approx^y_x u$ where $y = \bar{a}(x)$ and $z' = \bar{a}(z)$, we get $w \approx^{z'}_z u$ by **L5**, and so $\mathcal{M}_2, w, z \models \psi$ just in case $\mathcal{M}_2, u, z' \models \psi$ by the induction hypothesis. Thus we may conclude that $\mathcal{M}_2, w, z \not\models \psi$ for some $z < y$, completing the case.

Inductive Case ($\varphi = \boxed{\mathrm{F}}\psi$): The proof is analogous to the case for $\boxed{\mathrm{P}}\psi$, using $x < z$ and $y < z'$ in place of $z < x$ and $z' < y$. $\square$

T7 *Both* ***P1*** *and* ***P2*** *are valid over a two-dimensional frame* $\langle W, T, \leqslant\rangle$ *with* $W \neq \varnothing$ *if and only if* $|T| \leqslant 1$.

Proof. ($\Leftarrow$) Suppose $|T| \leqslant 1$. Since every two-dimensional model requires T to be nonempty (**D10**), we have $|T| = 1$; let $T = \{x\}$. Then $\boxed{\mathrm{F}}\varphi$ requires $\mathcal{M}, w, y \models \varphi$ for all y with $x < y$, and $\boxed{\mathrm{P}}\varphi$ requires $\mathcal{M}, w, y \models \varphi$ for all y with $y < x$; since $T = \{x\}$ contains no such y in either case, both $\boxed{\mathrm{F}}\varphi$ and $\boxed{\mathrm{P}}\varphi$ are vacuously true at $\mathcal{M}, w, x$ for every φ. Since $\triangle\varphi \coloneqq \boxed{\mathrm{P}}\varphi \wedge \varphi \wedge \boxed{\mathrm{F}}\varphi$, it follows that $\mathcal{M}, w, x \models \triangle\varphi$ just in case $\mathcal{M}, w, x \models \varphi$. Given that $\Box\varphi$ entails $\mathcal{M}, u, x \models \varphi$ for all $u \in W$, we have in particular $\mathcal{M}, w, x \models \varphi$, and so $\mathcal{M}, w, x \models \triangle\varphi$. Therefore $\Box\varphi \to \triangle\varphi$ is valid, and **P2** follows by contraposition as above.

($\Rightarrow$) Suppose $|T| \geqslant 2$. Since $\leqslant$ is a weak total order on T, there exist distinct $x, y \in T$ with $x < y$. Let $w \in W$ be arbitrary, and define $|p_0| = W \times \{x\}$. For any $u \in W$, we have $(u, x) \in |p_0|$, and so $\mathcal{M}, u, x \models p_0$. By the semantic clause for $\Box$, it follows that $\mathcal{M}, w, x \models \Box p_0$. However, since $x < y$ and $(w, y) \notin |p_0|$, we have $\mathcal{M}, w, y \not\models p_0$. Since $x < y$, it follows that $\mathcal{M}, w, x \not\models \boxed{\mathrm{F}}p_0$, and so $\mathcal{M}, w, x \not\models \triangle p_0$.

Therefore $\mathcal{M}, w, x \not\vDash \Box p_0 \to \triangle p_0$, and **P1** is falsified. Since **P2** is equivalent to **P1**, it follows that **P2** is also falsified. $\square$

T8 *Both **P1** and **P2** are valid over the abundant two-dimensional models of $\mathcal{L}^{\mathrm{K}}$.*

Proof. Let $\mathcal{M} = \langle W, T, \leqslant, |\cdot| \rangle$ be an abundant two-dimensional model of $\mathcal{L}^{\mathrm{K}}$. Assume for *reductio* that $\mathcal{M}, w, x \not\vDash \Box\varphi \to \triangle\varphi$ for some $w \in W$, $x \in T$, and well-formed sentence φ of $\mathcal{L}^{\mathrm{K}}$. Thus $\mathcal{M}, w, x \vDash \Box\varphi$ and $\mathcal{M}, w, x \not\vDash \triangle\varphi$.

By the semantic clause for $\Box$, we have $\mathcal{M}, u, x \vDash \varphi$ for all $u \in W$. In particular, $\mathcal{M}, w, x \vDash \varphi$. Given that $\triangle\varphi \coloneqq \boxed{\scriptstyle P}\varphi \wedge \varphi \wedge \boxed{\scriptstyle F}\varphi$ and $\mathcal{M}, w, x \vDash \varphi$, it follows from $\mathcal{M}, w, x \not\vDash \triangle\varphi$ that either $\mathcal{M}, w, x \not\vDash \boxed{\scriptstyle P}\varphi$ or $\mathcal{M}, w, x \not\vDash \boxed{\scriptstyle F}\varphi$.

Case 1: Assume $\mathcal{M}, w, x \not\vDash \boxed{\scriptstyle P}\varphi$. By the semantic clause for $\boxed{\scriptstyle P}$, there exists $y \in T$ with $y < x$ such that $\mathcal{M}, w, y \not\vDash \varphi$. Since $\mathcal{M}$ is abundant, there exists $w' \in W$ such that $w \approx_y^x w'$. By **L6**, $\mathcal{M}, w, y \vDash \varphi$ just in case $\mathcal{M}, w', x \vDash \varphi$. Since $\mathcal{M}, w, y \not\vDash \varphi$, we have $\mathcal{M}, w', x \not\vDash \varphi$. This contradicts that $\mathcal{M}, u, x \vDash \varphi$ for all $u \in W$.

Case 2: Assume $\mathcal{M}, w, x \not\vDash \boxed{\scriptstyle F}\varphi$. By the semantic clause for $\boxed{\scriptstyle F}$, there exists $z \in T$ with $x < z$ such that $\mathcal{M}, w, z \not\vDash \varphi$. Since $\mathcal{M}$ is abundant, there exists $w'' \in W$ such that $w \approx_z^x w''$. By **L6**, $\mathcal{M}, w, z \vDash \varphi$ just in case $\mathcal{M}, w'', x \vDash \varphi$. Since $\mathcal{M}, w, z \not\vDash \varphi$, we have $\mathcal{M}, w'', x \not\vDash \varphi$. This contradicts that $\mathcal{M}, u, x \vDash \varphi$ for all $u \in W$.

Both cases yield a contradiction. Therefore $\mathcal{M}, w, x \vDash \Box\varphi \to \triangle\varphi$ for all abundant models $\mathcal{M}$, worlds $w \in W$, times $x \in T$, and well-formed sentences φ of $\mathcal{L}^{\mathrm{K}}$. Hence **P1** is valid over the abundant two-dimensional models of $\mathcal{L}^{\mathrm{K}}$.

Since $\Box\neg\varphi \to \triangle\neg\varphi$ by **P1**, contraposition yields $\neg\triangle\neg\varphi \to \neg\Box\neg\varphi$. By definition, $\triangledown\varphi \to \Diamond\varphi$, and so **P2** is valid over the abundant two-dimensional models of $\mathcal{L}^{\mathrm{K}}$. $\square$

T9 *Abundant models with at least two distinct times are unbounded.*

Proof. Let $\mathcal{M} = \langle W, T, \leqslant, |\cdot| \rangle$ be an abundant two-dimensional model of $\mathcal{L}^{\mathrm{K}}$ where $x < y$ for $x, y \in T$. We prove that T is unbounded below and unbounded above.

Below: Assume for *reductio* that T is bounded below. Thus there is some $t_{\min} \in T$ where $t_{\min} \leqslant z$ for all $z \in T$. Since $t_{\min} \leqslant x < y$, either $t_{\min} < x$ or $t_{\min} = x < y$. In either case, there exists $t \in \{x, y\}$ with $t_{\min} < t$. Letting $w \in W$, it follows by abundance that there is some $w' \in W$ such that $w \approx_{t_{\min}}^t w'$. By definition, there is an order automorphism $\bar{a} : T \to T$ where $t = \bar{a}(t_{\min})$. Since $t_{\min} \leqslant z$ for all $z \in T$ and $\bar{a}$ is order-preserving, we have $\bar{a}(t_{\min}) \leqslant \bar{a}(z)$ for all $z \in T$. Since $\bar{a}$ is surjective, for any $z' \in T$ there exists $z \in T$ where $\bar{a}(z) = z'$. Thus $\bar{a}(t_{\min}) \leqslant z'$ for all $z' \in T$, so $\bar{a}(t_{\min})$ is minimal in T. Therefore $\bar{a}(t_{\min}) = t_{\min}$, and so $t = t_{\min}$, which contradicts $t_{\min} < t$. We may conclude by *reductio* that T is unbounded below.

Above: Assume for *reductio* that T is bounded above. Thus there is some $t_{\max} \in T$ where $z \leqslant t_{\max}$ for all $z \in T$. Since $x < y \leqslant t_{\max}$, either $y < t_{\max}$ or $x < y = t_{\max}$. In either case, there exists $t \in \{x, y\}$ with $t < t_{\max}$. Letting $w \in W$, it follows by abundance that there is some $w' \in W$ such that $w \approx_{t_{\max}}^t w'$. By definition, there is an order automorphism $\bar{a} : T \to T$ where $t = \bar{a}(t_{\max})$. Since $z \leqslant t_{\max}$ for all $z \in T$ and $\bar{a}$ is order-preserving, we have $\bar{a}(z) \leqslant \bar{a}(t_{\max})$ for all $z \in T$. Since $\bar{a}$ is surjective, for any $z' \in T$ there exists $z \in T$ where $\bar{a}(z) = z'$. Thus $z' \leqslant \bar{a}(t_{\max})$ for all $z' \in T$, so

$\bar{a}(t_{\max})$ is maximal in T. Therefore $\bar{a}(t_{\max}) = t_{\max}$, and so $t = t_{\max}$, which contradicts $t < t_{\max}$. We may conclude by *reductio* that T is unbounded above. $\square$

5.3 Task Semantics

We begin this section by restating a number of definitions for convenience:

D18 The language $\mathcal{L} \coloneqq \langle \mathbb{L}, \bot, \to, \Box, \boxed{\text{P}}, \boxed{\text{F}} \rangle$ where $\mathbb{L} \coloneqq \{p_i : i \in \mathbb{N}\}$ is a countable set of sentence letters and the remaining symbols denote falsity, material implication, the metaphysical necessity operator, the universal past tense operator, and the universal future tense operator, respectively.

D19 A *temporal order* is a nontrivial totally ordered abelian group $\mathcal{D} = \langle D, +, 0, \leqslant \rangle$ with *positive cone* $D^+ \coloneqq \{x \in D : x \geqslant 0\}$.

D20 A *task relation* on a nonempty set of *world states* W over a temporal order $\mathcal{D}$ is any parameterized relation $w \Rightarrow_x u$ for $w, u \in W$ and $x \in D^+$, extended to negative durations by the *converse convention* $w \Rightarrow_{-x} u \coloneqq u \Rightarrow_x w$ for $x \geqslant 0$, determining the following for any world states $w, v \in W$ and durations $x, y \in D$:

Fiber: $\texttt{fib}(w, x) \coloneqq \{u \in W : w \Rightarrow_x u\}$.

Cone: $(w)_x \coloneqq \bigcup\limits_{|y|<x} \texttt{fib}(w, y)$ where $x > 0$.

Segment: $[w, v]_x^y \coloneqq \texttt{fib}(w, x) \cap \texttt{fib}(v, -y)$ where $x, y \geqslant 0$.

D21 A nonempty family of sets $\mathcal{S}$ is *directed* just in case $S \subseteq S_1 \cap S_2$ for some $S \in \mathcal{S}$ whenever $S_1, S_2 \in \mathcal{S}$.

D22 A *frame* is any $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ where W is a nonempty set of world states, $\mathcal{D}$ is a temporal order, and $\Rightarrow$ is a task relation satisfying the following for $x, y \geqslant 0$:

Compositionality: $w \Rightarrow_{x+y} v$ if and only if $w \Rightarrow_x u$ and $u \Rightarrow_y v$ for some $u \in W$.

Seriality: $w \Rightarrow_x u$ and $v \Rightarrow_x w$ for some $u, v \in W$.

Limit: $\bigcap\limits_{x>0} (w)_x = \{w\}$.

Spherical: $\bigcap \mathcal{S} \neq \varnothing$ for any directed family $\mathcal{S}$ of nonempty fibers and segments.

D23 Given a frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$, define:

Basic Opens: $B_{\mathcal{F}} \coloneqq \{(w)_x : w \in W \text{ and } x \in D \text{ with } x > 0\}$.

Topology: $\mathcal{T}_{\mathcal{F}} \coloneqq \langle W, \mathcal{O}_{\mathcal{F}} \rangle$ where $\mathcal{O}_{\mathcal{F}}$ is the result of closing $B_{\mathcal{F}}$ under arbitrary union and finite intersection.

Discrete: A topology is *discrete* just in case every subset of W is open.

Closure: $\overline{S} \coloneqq \{w \in W : O \cap S \neq \varnothing$ for every open $O \in \mathcal{T}_{\mathcal{F}}$ where $w \in O\}$ for $S \subseteq W$.

T1: A topology is *T1* just in case $\overline{\{w\}} = \{w\}$ for all $w \in W$.

R0: A topology is *R0* just in case $w \in \overline{\{u\}}$ iff $u \in \overline{\{w\}}$ for all $w, u \in W$.

L7 *$w \Rightarrow_0 w$ for every world state $w \in W$ in every frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$.*

Proof. By *Seriality* at $x = 0$, there is some $u \in W$ where $w \Rightarrow_0 u$. Since $|0| < x$ for every $x > 0$, it follows that $u \in (w)_x$ for every $x > 0$. Thus $u \in \bigcap_{x>0}(w)_x = \{w\}$ by *Limit*, and so $u = w$, whence $w \Rightarrow_0 w$. □

T10 *$\mathcal{T}_{\mathcal{F}}$ is T1 for every frame $\mathcal{F}$.*

Proof. Let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ be a frame and $u \in W$. Since $u \in O \cap \{u\}$ for every open $O \in \mathcal{O}_{\mathcal{F}}$ containing u, it follows by *Closure* in **D23** that $\{u\} \subseteq \overline{\{u\}}$.

For the converse inclusion, suppose $w \in \overline{\{u\}}$, so that $u \in O$ for every open $O \in \mathcal{O}_{\mathcal{F}}$ containing w by **Closure**. Since $w \Rightarrow_0 w$ by **L7**, each basic open $(w)_x$ with $x > 0$ contains w, and so $u \in (w)_x$ by the above. Thus for every $x > 0$, there is some $y \in D$ where $|y| < x$ and $w \Rightarrow_y u$, and so $u \Rightarrow_{-y} w$ by the converse convention, whence $w \in (u)_x$ for every $x > 0$. Thus $w \in \bigcap_{x>0}(u)_x = \{u\}$ by *Limit*, and so $w = u$, establishing $\overline{\{u\}} \subseteq \{u\}$. Hence $\overline{\{u\}} = \{u\}$ for every $u \in W$, and so $\mathcal{T}_{\mathcal{F}}$ is *T1* by **D23**. □

C3 *$\mathcal{T}_{\mathcal{F}}$ is R0 for every frame $\mathcal{F}$.*

Proof. Given $w \in \overline{\{u\}}$, **T10** yields $\overline{\{u\}} = \{u\}$, so $w = u$, whence $u \in \overline{\{w\}}$ by **T10** again. The converse direction is symmetric, and so $\mathcal{T}_{\mathcal{F}}$ is *R0* by **D23**. □

D24 A *partial history* over a frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ is a function $\tau : X \to W$ on a nonempty set $X \subseteq D$ where $\tau(x) \Rightarrow_{y-x} \tau(y)$ for all times $x, y \in X$. A *world history* is any partial history whose domain X is *convex*, so that $y \in X$ whenever $x, z \in X$ and $x < y < z$. A world history is *total*— equivalently, a *possible world*— just in case $X = D$. A partial history σ *extends* τ just in case $\mathtt{dom}(\tau) \subseteq \mathtt{dom}(\sigma)$ and $\tau(x) = \sigma(x)$ for all $x \in \mathtt{dom}(\tau)$. The set of all total world histories over $\mathcal{F}$ is denoted $H_{\mathcal{F}}$.

D25 For a partial history $\tau : X \to W$ over a frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ and duration $z \in D\backslash X$, the *constraints on* z are the segments $[\tau(t), \tau(s)]^{s-z}_{z-t}$ for times $t, s \in X$ where $t < z < s$ when both $t, s \in X$, and the fibers $\mathtt{fib}(\tau(t), z - t)$ for $t \in X$ otherwise.

L8 *For any partial history $\tau : X \to W$ over a frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ and duration $z \in D\backslash X$, the fibers $\mathtt{fib}(\tau(t'), z-t') \subseteq \mathtt{fib}(\tau(t), z-t)$ nest for all times $t \leqslant t' < z$ in X and symmetrically for all times $z < t' \leqslant t$ in X, while the segments $[\tau(t'), \tau(s')]^{s'-z}_{z-t'} \subseteq [\tau(t), \tau(s)]^{s-z}_{z-t}$ nest for all times $t \leqslant t' < z < s' \leqslant s$ in X.*

Proof. Suppose $t \leqslant t' < z$ in X and consider any $u \in \mathtt{fib}(\tau(t'), z - t')$, so that $\tau(t') \Rightarrow_{z-t'} u$. Since τ is a partial history, $\tau(t) \Rightarrow_{t'-t} \tau(t')$, and so $\tau(t) \Rightarrow_{z-t} u$

by *Compositionality*, where $z - t = (t' - t) + (z - t')$, whence $u \in \mathtt{fib}(\tau(t), z - t)$. Suppose instead $z < t' \leqslant t$ in X and consider any $u \in \mathtt{fib}(\tau(t'), z - t')$, so that $u \Rightarrow_{t'-z} \tau(t')$ by the converse convention. Since $\tau(t') \Rightarrow_{t-t'} \tau(t)$, these constraints compose by *Compositionality* to give $u \Rightarrow_{t-z} \tau(t)$, whence $u \in \mathtt{fib}(\tau(t), z - t)$ by the converse convention again. Finally, for times $t \leqslant t' < z < s' \leqslant s$ in X, the segment $[\tau(t'), \tau(s')]_{z-t'}^{s'-z} = \mathtt{fib}(\tau(t'), z - t') \cap \mathtt{fib}(\tau(s'), z - s')$ is included factorwise in $[\tau(t), \tau(s)]_{z-t}^{s-z} = \mathtt{fib}(\tau(t), z - t) \cap \mathtt{fib}(\tau(s), z - s)$ by the two inclusions above. $\square$

L9 *For any partial history $\tau : X \to W$ over a frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ and duration $z \in D \backslash X$, every constraint imposed on z is nonempty.*

Proof. A fiber $\mathtt{fib}(\tau(t), z - t)$ imposed on z is nonempty by *Seriality*: when $t < z$, some $u \in W$ satisfies $\tau(t) \Rightarrow_{z-t} u$, and when $t > z$, some $u \in W$ satisfies $u \Rightarrow_{t-z} \tau(t)$, which is $\tau(t) \Rightarrow_{z-t} u$ by the converse convention. A segment $[\tau(t), \tau(s)]_{z-t}^{s-z}$ imposed on z for times $t < z < s$ is nonempty by *Compositionality*: since τ is a partial history, $\tau(t) \Rightarrow_{s-t} \tau(s)$ where $s - t = (z - t) + (s - z)$, and so *Compositionality* provides some $u \in W$ where $\tau(t) \Rightarrow_{z-t} u$ and $u \Rightarrow_{s-z} \tau(s)$, whence $u \in [\tau(t), \tau(s)]_{z-t}^{s-z}$. $\square$

L10 *For any partial history $\tau : X \to W$ over a frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ and duration $z \in D \backslash X$, the constraints imposed on z form a directed family of nonempty sets.*

Proof. The family is nonempty since X is nonempty, and every member is nonempty by **L9**, so it remains to establish directedness. Split X into $A \coloneqq \{t \in X : t < z\}$ and $C \coloneqq \{s \in X : s > z\}$, so that by **D25** the constraints imposed on z are the segments $S_{t,s} \coloneqq [\tau(t), \tau(s)]_{z-t}^{s-z}$ for pairs $\langle t, s \rangle \in A \times C$ when A and C are both nonempty, and the fibers $F_t \coloneqq \mathtt{fib}(\tau(t), z - t)$ for $t \in X$ otherwise. For any two segments $S_{t,s}$ and $S_{t',s'}$, the times $t'' \coloneqq \max\{t, t'\}$ and $s'' \coloneqq \min\{s, s'\}$ belong to A and C respectively, so $S_{t'',s''}$ is a member of the family where $S_{t'',s''} \subseteq S_{t,s} \cap S_{t',s'}$ by **L8**. For any two fibers F_t and $F_{t'}$, the times t and t' lie on the same side of z, and so the fiber imposed by the time nearer to z is included in the other by **L8**, and hence in $F_t \cap F_{t'}$. $\square$

L11 *For any partial history $\tau : X \to W$ over a frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ and duration $z \in D \backslash X$, a world state $u \in W$ belongs to every member of the constraints imposed on z just in case $\tau(t) \Rightarrow_{z-t} u$ for every $t \in X$.*

Proof. Since $\tau(t) \Rightarrow_{z-t} u$ asserts that $u \in \mathtt{fib}(\tau(t), z - t)$, it suffices to show that u belongs to every member of the constraints imposed on z just in case $u \in \mathtt{fib}(\tau(t), z - t)$ for every $t \in X$. When the assignments lie on one side of z, the constraints imposed on z are exactly these fibers by **D25**. When assignments flank z, the constraints are instead the segments $[\tau(t), \tau(s)]_{z-t}^{s-z} = \mathtt{fib}(\tau(t), z - t) \cap \mathtt{fib}(\tau(s), z - s)$ for times $t, s \in X$ where $t < z < s$ by **D25**.

If $u \in \mathtt{fib}(\tau(r), z - r)$ for every $r \in X$, then for any times $t, s \in X$ where $t < z < s$, both factors $\mathtt{fib}(\tau(t), z - t)$ and $\mathtt{fib}(\tau(s), z - s)$ of the segment $[\tau(t), \tau(s)]_{z-t}^{s-z}$ contain u, and so u belongs to every segment of the family.

Conversely, suppose u belongs to every segment of the family, and consider any $t \in X$. If $t < z$, there is some $s \in X$ where $s > z$ since assignments flank z, and the segment $[\tau(t), \tau(s)]^{s-z}_{z-t}$ is included in its factor $\texttt{fib}(\tau(t), z-t)$, so $u \in \texttt{fib}(\tau(t), z-t)$. If instead $t > z$, there is some $s \in X$ where $s < z$, and the segment $[\tau(s), \tau(t)]^{t-z}_{z-s}$ is included in its factor $\texttt{fib}(\tau(t), z-t)$ just the same. □

L12 *For any partial history $\tau : X \to W$ over a frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ and duration $z \in D\backslash X$, the function $\tau \cup \{\langle z, u\rangle\}$ is a partial history on $X \cup \{z\}$ just in case u belongs to every member of the constraints imposed on z.*

Proof. The function $\tau \cup \{\langle z, u\rangle\}$ is a partial history just in case $\tau(t) \Rightarrow_{z-t} u$ for every $t \in X$. Here the constraints among the times in X are those of τ, where the instance at z itself is the zero loop $u \Rightarrow_0 u$ of **L7**. By **L11**, these conditions hold just in case u belongs to every member of the constraints imposed on z. □

L13 *Every partial history $\tau : X \to W$ over a frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ extends to a partial history on $X \cup \{z\}$ for any duration $z \in D$.*

Proof. If $z \in X$, then τ extends itself. Otherwise the constraints imposed on z form a directed family of nonempty fibers and segments by **L10**, and so *Spherical* provides some $u \in W$ belonging to every member, whence $\tau \cup \{\langle z, u\rangle\}$ is a partial history on $X \cup \{z\}$ extending τ by **L12**. When the family has a $\subseteq$-least member— as **L8** provides whenever X contains an assignment nearest to z on each side it occupies— that member already contains a candidate and *Spherical* is not needed. □

C4 *Every frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ with finite W satisfies Spherical, choice-free.*

Proof. Let $\mathcal{S}$ be a directed family of nonempty fibers and segments, as in **D21**. Since W is finite, every member of $\mathcal{S}$ is a subset of W, so $\mathcal{S}$ has finitely many distinct members. Directedness gives, for any two members of $\mathcal{S}$, some member of $\mathcal{S}$ contained in their intersection, and iterating this over the finitely many members yields a least member $S^* \in \mathcal{S}$ with $S^* \subseteq S$ for every $S \in \mathcal{S}$. Since S^* is itself a member of $\mathcal{S}$, it is nonempty, and $S^* \subseteq \bigcap \mathcal{S} \subseteq S^*$, so $\bigcap \mathcal{S} = S^* \neq \varnothing$. □

T11 *Every partial history $\tau : X \to W$ over a frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ is extended by some total world history $\sigma \in H_{\mathcal{F}}$.*[77]

Proof. The partial histories extending τ are partially ordered by extension, and every chain among them is bounded above by its union, which restricts on any pair of times to a single member of the chain and so is itself a partial history. By Zorn's lemma, there is a maximal partial history $\sigma : T \to W$ extending τ. If $T \neq D$, then σ extends to a partial history on $T \cup \{z\}$ for any $z \in D\backslash T$ by **L13**, contradicting maximality. Thus $T = D$, whence $\sigma \in H_{\mathcal{F}}$ is a total world history extending τ by **D24**. □

[77]The proof appeals to Zorn's lemma and hence to the axiom of choice, and so the derivation of *Occurrence* from *Seriality* and *Spherical* in **C5** is a theorem of ZFC, in contrast with the choice-free derivation of the zero loops in **L7** and of *Spherical* for finite W in **C4**.

C5 *For any frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$, world state $w \in W$, and time $x \in D$, there is a total world history $\tau \in H_{\mathcal{F}}$ where $\tau(x) = w$, and so $H_{\mathcal{F}} \neq \varnothing$.*

Proof. Since $w \Rightarrow_0 w$ by **L7**, the function $\{\langle x, w \rangle\}$ is a partial history on the nonempty set $\{x\}$. By **T11**, some total world history $\tau \in H_{\mathcal{F}}$ extends it, where $\tau(x) = w$. Since W is nonempty, $H_{\mathcal{F}} \neq \varnothing$. □

D26 A *model* of $\mathcal{L}$ is a structure $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ where $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ is a frame and $|p_i| \subseteq W$ for every sentence letter $p_i \in \mathbb{L}$. Relative to a model $\mathcal{M}$, possible world $\tau \in H_{\mathcal{F}}$, and time $x \in D$, *truth* is defined recursively as follows:

(p_i) $\mathcal{M}, \tau, x \vDash p_i$ *iff* $\tau(x) \in |p_i|$.

($\bot$) $\mathcal{M}, \tau, x \nvDash \bot$.

($\to$) $\mathcal{M}, \tau, x \vDash \varphi \to \psi$ *iff* $\mathcal{M}, \tau, x \nvDash \varphi$ or $\mathcal{M}, \tau, x \vDash \psi$.

($\Box$) $\mathcal{M}, \tau, x \vDash \Box\varphi$ *iff* $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in H_{\mathcal{F}}$.

($\boxed{\mathrm{P}}$) $\mathcal{M}, \tau, x \vDash \boxed{\mathrm{P}}\varphi$ *iff* $\mathcal{M}, \tau, y \vDash \varphi$ for all $y \in D$ where $y < x$.

($\boxed{\mathrm{F}}$) $\mathcal{M}, \tau, x \vDash \boxed{\mathrm{F}}\varphi$ *iff* $\mathcal{M}, \tau, y \vDash \varphi$ for all $y \in D$ where $x < y$.

D27 For a frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$, the possible worlds $\tau, \sigma \in H_{\mathcal{F}}$ are *time-shifted from x to y*— written $\tau \approx_x^y \sigma$— *iff* there exists a *translation* $\bar{a} : D \to D$, $\bar{a}(z) = z + d$ for some $d \in D$, where $y = \bar{a}(x)$ and $\tau(z) = \sigma(\bar{a}(z))$ for all $z \in D$.

L14 *For any frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$, possible world $\tau \in H_{\mathcal{F}}$, and times $x, y \in D$, there is a possible world $\sigma \in H_{\mathcal{F}}$ where $\tau \approx_x^y \sigma$ is witnessed by $\bar{a}(z) = z - x + y$.*

Proof. Let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ be a frame, $\tau \in H_{\mathcal{F}}$ a possible world where $x, y \in D$ are arbitrary times. Define $\bar{a} : D \to D$ by $\bar{a}(z) = z - x + y$.

Since $\mathcal{D} = \langle D, +, 0, \leqslant \rangle$ is an abelian group, $\bar{a}$ is a bijection. Supposing $z_1 \leqslant z_2$, it follows that $z_1 - x + y \leqslant z_2 - x + y$, and so $\bar{a}(z_1) \leqslant \bar{a}(z_2)$. Thus $\bar{a}$ is an order automorphism with $y = \bar{a}(x)$ where the inverse is $\bar{a}^{-1}(z) = z + x - y$.

Define $\sigma(z) = \tau(\bar{a}^{-1}(z))$ for all $z \in D$. Since $\bar{a}$ is a bijection of D, the function σ is total. Suppose $z_1, z_2 \in D$. Then:

$$\begin{aligned} \bar{a}^{-1}(z_2) - \bar{a}^{-1}(z_1) &= (z_2 + x - y) - (z_1 + x - y) \\ &= z_2 - z_1. \end{aligned}$$

Since τ is a possible world, $\tau(\bar{a}^{-1}(z_1)) \Rightarrow_{z_2 - z_1} \tau(\bar{a}^{-1}(z_2))$, and so $\sigma(z_1) \Rightarrow_{z_2 - z_1} \sigma(z_2)$. Thus $\sigma \in H_{\mathcal{F}}$ and $\tau \approx_x^y \sigma$ is witnessed by $\bar{a}$. □

L15 *$\mathcal{M}, \tau, x \vDash \varphi$ just in case $\mathcal{M}, \sigma, y \vDash \varphi$ for any model $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ of $\mathcal{L}$, well-formed sentence φ of $\mathcal{L}$, and possible worlds $\tau, \sigma \in H_{\mathcal{F}}$ where $\tau \approx_x^y \sigma$ is witnessed by the time-shift function $\bar{a}(z) = z - x + y$.*

Proof. We proceed by induction on the complexity of φ.

Base Case (p_i): The first and third biconditionals follow from **D26**.

$$\begin{aligned}\mathcal{M}, \tau, x \vDash p_i &\Leftrightarrow \tau(x) \in |p_i| \\ &\Leftrightarrow \sigma(y) \in |p_i| \\ &\Leftrightarrow \mathcal{M}, \sigma, y \vDash p_i\end{aligned}$$

For the second biconditional, $\tau(x) = \sigma(\bar{a}(x)) = \sigma(y)$ by **D27**.

Base Case ($\bot$): By **D26**, $\mathcal{M}, \tau, x \nvDash \bot$ and $\mathcal{M}, \sigma, y \nvDash \bot$.

Inductive Case ($\varphi \to \psi$): The first and third biconditionals follow from **D26** and the second biconditional follows from the inductive hypothesis.

$$\begin{aligned}\mathcal{M}, \tau, x \vDash \varphi \to \psi &\Leftrightarrow \mathcal{M}, \tau, x \nvDash \varphi \text{ or } \mathcal{M}, \tau, x \vDash \psi \\ &\Leftrightarrow \mathcal{M}, \sigma, y \nvDash \varphi \text{ or } \mathcal{M}, \sigma, y \vDash \psi \\ &\Leftrightarrow \mathcal{M}, \sigma, y \vDash \varphi \to \psi\end{aligned}$$

Inductive Case ($\Box\varphi$): The first and third biconditionals follow from **D26**.

$$\begin{aligned}\mathcal{M}, \tau, x \nvDash \Box\varphi &\Leftrightarrow \mathcal{M}, \rho, x \nvDash \varphi \text{ for some } \rho \in H_{\mathcal{F}} \\ &\Leftrightarrow \mathcal{M}, \rho', y \nvDash \varphi \text{ for some } \rho' \in H_{\mathcal{F}} \\ &\Leftrightarrow \mathcal{M}, \sigma, y \nvDash \Box\varphi\end{aligned}$$

To justify the forward direction of the second biconditional, assume $\rho \in H_{\mathcal{F}}$ with $\mathcal{M}, \rho, x \nvDash \varphi$. By **L14** there is $\rho' \in H_{\mathcal{F}}$ where $\rho \approx^y_x \rho'$ is witnessed by $\bar{a}(z) = z - x + y$, so $\mathcal{M}, \rho', y \nvDash \varphi$ by the inductive hypothesis. Now assume $\rho' \in H_{\mathcal{F}}$ with $\mathcal{M}, \rho', y \nvDash \varphi$. By **L14**, there is $\rho \in H_{\mathcal{F}}$ where $\rho' \approx^x_y \rho$ is witnessed by $\bar{b}(z) = z - y + x$, so by the inductive hypothesis $\mathcal{M}, \rho, x \nvDash \varphi$.

Inductive Case ($\boxed{\text{P}}\varphi$): The first and third biconditionals follow from **D26**.

$$\begin{aligned}\mathcal{M}, \tau, x \nvDash \boxed{\text{P}}\varphi &\Leftrightarrow \mathcal{M}, \tau, x' \nvDash \varphi \text{ for some } x' \in D \text{ where } x' < x \\ &\Leftrightarrow \mathcal{M}, \sigma, y' \nvDash \varphi \text{ for some } y' \in D \text{ where } y' < y \\ &\Leftrightarrow \mathcal{M}, \sigma, y \nvDash \boxed{\text{P}}\varphi\end{aligned}$$

To justify the forward direction of the second biconditional, assume $\mathcal{M}, \tau, x' \nvDash \varphi$ for some $x' < x$. Since $\bar{a}$ is an order automorphism, we have $\bar{a}(x') < \bar{a}(x)$. Letting $y' = \bar{a}(x')$, it follows that $y' < y$. By assumption $\tau \approx^y_x \sigma$, and so $\tau(z) = \sigma(\bar{a}(z))$ for all $z \in D$ by **D27**. Given that $\bar{a}(x') = y'$, the same $\bar{a}$ that witnesses $\tau \approx^y_x \sigma$ also witnesses $\tau \approx^{y'}_{x'} \sigma$. Thus $\mathcal{M}, \sigma, y' \nvDash \varphi$ by hypothesis.

Now assume $\mathcal{M}, \sigma, y' \nvDash \varphi$ for some $y' < y$. Since $\bar{a}$ is a bijection, there exists x' with $\bar{a}(x') = y'$. Since $\bar{a}$ is an order automorphism and $\bar{a}(x') = y' < y = \bar{a}(x)$, it follows that $x' < x$. It follows by the same reasoning above that $\bar{a}$ witnesses $\tau \approx^{y'}_{x'} \sigma$, and so by the inductive hypothesis $\mathcal{M}, \tau, x' \nvDash \varphi$.

Inductive Case ($\boxed{\text{F}}\varphi$): The proof is similar to the $\boxed{\text{P}}$ case. $\square$

D28 A task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ is:

DISCRETE if for any $x \in D$, whenever there exists $y > x$, there is a least such $y' > x$ satisfying $z \geqslant y'$ for all $z > x$.

DENSE if for any $x, y \in D$ where $x < y$, there exists $z \in D$ where $x < z < y$.

COMPLETE if every nonempty $S \subseteq D$ bounded above has a least upper bound in D.

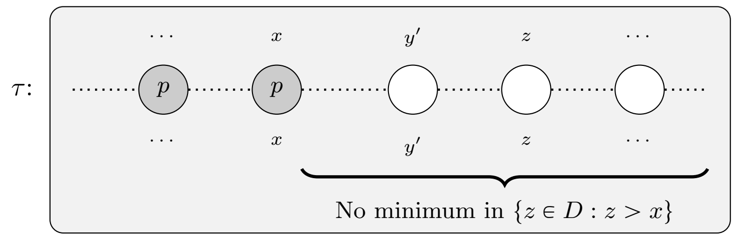
DETERMINISTIC if $u = v$ whenever $w \Rightarrow_x u$ and $w \Rightarrow_x v$ for $w, u, v \in W$ and $x \in D$.

D29 A well-formed sentence φ of $\mathcal{L}$ is *valid over a frame* $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ which we may write $\vDash_{\mathcal{F}} \varphi$ if and only if $\mathcal{M}, \tau, x \vDash \varphi$ for every model $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ where $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$, possible world $\tau \in H_{\mathcal{F}}$, and time $x \in D$.[78]

T12 *$\mathcal{F} \vDash (\boxed{\text{P}}\varphi \wedge \varphi \wedge \Diamond_{\text{F}}\top) \to \Diamond_{\text{F}}\boxed{\text{P}}\varphi$ iff $\mathcal{F}$ is a* DISCRETE *frame.*

Proof. ($\Leftarrow$) Suppose $\mathcal{D}$ is DISCRETE and let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ be an arbitrary frame. Let $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ be a model with $\tau \in H_{\mathcal{F}}$ and $x \in D$ where $\mathcal{M}, \tau, x \vDash \boxed{\text{P}}\varphi \wedge \varphi \wedge \Diamond_{\text{F}}\top$. From $\Diamond_{\text{F}}\top$, there exists some $y > x$. By **D28**, there is a least $x^+ > x$ where $z \geqslant x^+$ for all $z > x$. Let $t < x^+$ be arbitrary. Since x^+ is the least time greater than x, we know $t \leqslant x$. If $t < x$, then $\mathcal{M}, \tau, t \vDash \varphi$ by $\boxed{\text{P}}\varphi$. If $t = x$, then $\mathcal{M}, \tau, t \vDash \varphi$ directly. Thus $\mathcal{M}, \tau, x^+ \vDash \boxed{\text{P}}\varphi$, giving $\mathcal{M}, \tau, x \vDash \Diamond_{\text{F}}\boxed{\text{P}}\varphi$. Since $\mathcal{F}$, $\mathcal{M}$, τ, and x were arbitrary, $(\boxed{\text{P}}\varphi \wedge \varphi \wedge \Diamond_{\text{F}}\top) \to \Diamond_{\text{F}}\boxed{\text{P}}\varphi$ is valid over every frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$.

($\Rightarrow$) Suppose $\mathcal{D}$ is not DISCRETE. Then there exists $x \in D$ such that $\{z \in D : z > x\}$ is nonempty but has no minimum. Let $W = D$ with $w \Rightarrow_d u$ *iff* $u = w + d$ for $d \geqslant 0$, let $\mathcal{F}' = \langle W, \mathcal{D}, \Rightarrow\rangle$, and define $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ with $|p| = \{w \in W : w \leqslant x\}$ and $\tau : D \to W$ by $\tau(y) = y$. Both directions of *Compositionality* are immediate with $u = w + x$ the unique intermediate state for $w \Rightarrow_{x+y} v$. *Seriality* holds since the forward fiber of w at d is $\{w + d\}$ and the backward fiber is $\{w - d\}$, and since $(w)_d = \{u \in W : |u - w| < d\}$, we have $\bigcap_{d>0}(w)_d = \{w\}$ for *Limit*. *Spherical* holds since every fiber and nonempty segment is a singleton, so any two members of a directed family of nonempty fibers and segments include a common singleton member and so coincide, leaving one singleton whose intersection is nonempty. The total world histories over $\mathcal{F}'$ are the translations $\sigma_c(z) = z + c$, and in particular $\tau = \sigma_0 \in H_{\mathcal{F}'}$.



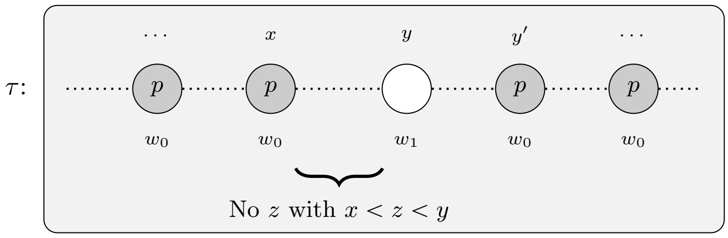
[78]Since $H_{\mathcal{F}} \neq \varnothing$ for every frame by **C5**, frame validity is never vacuous: every frame contributes evaluation points, and so $\nvDash_{\mathcal{F}} \bot$ for every frame $\mathcal{F}$.

Thus $\mathcal{M}, \tau, x \vDash p$ since $\tau(x) = x \in |p|$, and $\mathcal{M}, \tau, x \vDash \boxed{\scriptstyle\mathrm{P}}p$ since $\tau(y) = y \in |p|$ for all $y < x$ by **D26**. Since $\{z \in D : z > x\}$ is nonempty, $\mathcal{M}, \tau, x \vDash \Diamond_{\mathrm{F}}\top$. Suppose for contradiction that $\mathcal{M}, \tau, x \vDash \Diamond_{\mathrm{F}}\boxed{\scriptstyle\mathrm{P}}p$. Then there exists $z > x$ with $\mathcal{M}, \tau, z \vDash \boxed{\scriptstyle\mathrm{P}}p$. Since there is no minimum in $\{z' \in D : z' > x\}$, there exists $y' \in D$ with $x < y' < z$. But $\tau(y') = y' \notin |p|$ since $y' > x$, so $\mathcal{M}, \tau, y' \nvDash p$ by **D26**, contradicting $\mathcal{M}, \tau, z \vDash \boxed{\scriptstyle\mathrm{P}}p$. Therefore $\mathcal{M}, \tau, x \nvDash \Diamond_{\mathrm{F}}\boxed{\scriptstyle\mathrm{P}}p$, and so $\nvDash_{\mathcal{F}'} (\boxed{\scriptstyle\mathrm{P}}\varphi \wedge \varphi \wedge \Diamond_{\mathrm{F}}\top) \rightarrow \Diamond_{\mathrm{F}}\boxed{\scriptstyle\mathrm{P}}\varphi$, and the axiom is not valid over every frame with temporal order $\mathcal{D}$. $\square$

T13 $\mathcal{F} \vDash \boxed{\scriptstyle\mathrm{F}}\boxed{\scriptstyle\mathrm{F}}\varphi \rightarrow \boxed{\scriptstyle\mathrm{F}}\varphi$ *iff* $\mathcal{F}$ *is a* DENSE *frame.*

Proof. ($\Leftarrow$) Suppose $\mathcal{D}$ is DENSE and let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ be an arbitrary frame. Let $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ be a model with $\tau \in H_{\mathcal{F}}$ and $x \in D$ where $\mathcal{M}, \tau, x \vDash \boxed{\scriptstyle\mathrm{F}}\boxed{\scriptstyle\mathrm{F}}\varphi$. Assume for contradiction that $\mathcal{M}, \tau, x \nvDash \boxed{\scriptstyle\mathrm{F}}\varphi$. Then there is some $y > x$ with $\mathcal{M}, \tau, y \nvDash \varphi$ by **D26**. Since $\mathcal{D}$ is DENSE and $x < y$, there is a $z \in D$ where $x < z < y$ by **D28**. Since $\mathcal{M}, \tau, x \vDash \boxed{\scriptstyle\mathrm{F}}\boxed{\scriptstyle\mathrm{F}}\varphi$ and $z > x$, we have $\mathcal{M}, \tau, z \vDash \boxed{\scriptstyle\mathrm{F}}\varphi$ by **D26**. Thus we have $\mathcal{M}, \tau, y \vDash \varphi$ since $y > z$, contradicting $\mathcal{M}, \tau, y \nvDash \varphi$. Since $\mathcal{F}$, $\mathcal{M}$, τ, and x were arbitrary, $\boxed{\scriptstyle\mathrm{F}}\boxed{\scriptstyle\mathrm{F}}\varphi \rightarrow \boxed{\scriptstyle\mathrm{F}}\varphi$ is valid over every frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$.

($\Rightarrow$) Suppose $\mathcal{D}$ is not DENSE. Then there exist $x, y \in D$ with $x < y$ and no $z \in D$ satisfying $x < z < y$. Thus $\epsilon \coloneqq y - x$ is the least positive duration in D, since any positive $d < \epsilon$ would give $x < x + d < y$. Let $W = \{w_0, w_1\}$ where $w \Rightarrow_0 w'$ *iff* $w = w'$ and $w \Rightarrow_d w'$ for all $w, w' \in W$ and $d > 0$, let $\mathcal{F}' = \langle W, \mathcal{D}, \Rightarrow \rangle$, and define $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ with $|p| = \{w_0\}$ and $\tau : D \rightarrow W$ by $\tau(y) = w_1$ and $\tau(z) = w_0$ for all $z \neq y$. Both directions of *Compositionality* are immediate, taking $u = w$ or $u = w'$ for the factoring direction when a summand is zero and any $u \in W$ otherwise, *Seriality* holds since every fiber at a positive duration is all of W while every state loops at duration zero, and $(w)_\epsilon = \{w\}$ gives $\bigcap_{d>0}(w)_d = \{w\}$ for *Limit*. *Spherical* holds by **C4** since W is finite. Since every non-zero difference between times is covered by the relation, every function from D to W is a total world history over $\mathcal{F}'$, and so $\tau \in H_{\mathcal{F}'}$.



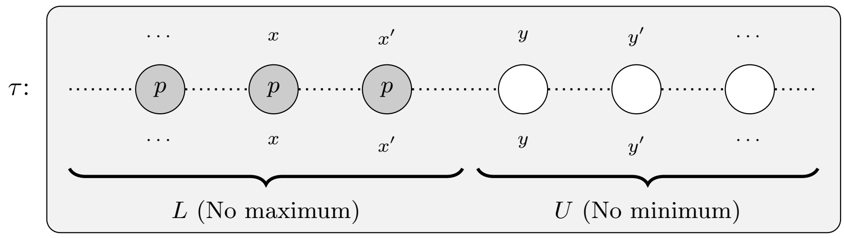
Given that $y > x$ and $\tau(y) = w_1 \notin |p|$, we have $\mathcal{M}, \tau, x \nvDash \boxed{\scriptstyle\mathrm{F}}p$ by **D26**. Let $y' > x$ be arbitrary. Since there is no z with $x < z < y$, it follows that $y' \geqslant y$. Let $z' > y'$ be arbitrary. Then $z' > y$, so $z' \neq y$ and $\tau(z') = w_0 \in |p|$, giving $\mathcal{M}, \tau, z' \vDash p$. Since $z' > y'$ was arbitrary, $\mathcal{M}, \tau, y' \vDash \boxed{\scriptstyle\mathrm{F}}p$. Since $y' > x$ was arbitrary, $\mathcal{M}, \tau, x \vDash \boxed{\scriptstyle\mathrm{F}}\boxed{\scriptstyle\mathrm{F}}p$ by **D26**. Therefore $\mathcal{M}, \tau, x \nvDash \boxed{\scriptstyle\mathrm{F}}\boxed{\scriptstyle\mathrm{F}}p \rightarrow \boxed{\scriptstyle\mathrm{F}}p$, and so $\nvDash_{\mathcal{F}'} \boxed{\scriptstyle\mathrm{F}}\boxed{\scriptstyle\mathrm{F}}\varphi \rightarrow \boxed{\scriptstyle\mathrm{F}}\varphi$, and the axiom is not valid over every frame with temporal order $\mathcal{D}$. $\square$

T14 $\mathcal{F} \vDash \triangle(\boxed{\scriptstyle\mathrm{P}}\varphi \rightarrow \Diamond_{\mathrm{F}}\boxed{\scriptstyle\mathrm{P}}\varphi) \rightarrow (\boxed{\scriptstyle\mathrm{P}}\varphi \rightarrow \boxed{\scriptstyle\mathrm{F}}\varphi)$ *iff* $\mathcal{F}$ *is a* COMPLETE *frame.*

Proof. ($\Leftarrow$) Suppose $\mathcal{D}$ is Complete and let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ be an arbitrary frame. Let $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ be a model with $\mathcal{M}, \tau, x \vDash \triangle(\boxed{P}\varphi \rightarrow \Diamond\boxed{P}\varphi)$ and $\mathcal{M}, \tau, x \vDash \boxed{P}\varphi$ for $\tau \in H_{\mathcal{F}}$ and $x \in D$. Suppose for contradiction that $\mathcal{M}, \tau, x \nvDash \boxed{F}\varphi$. Then there exists some $y > x$ with $\mathcal{M}, \tau, y \nvDash \varphi$. Let $B = \{z \in D : z > x \text{ and } \mathcal{M}, \tau, z \nvDash \varphi\}$, which is nonempty. Since B is bounded below by x, the set $\{-z : z \in B\}$ is bounded above by $-x$, so by **D28** it has a least upper bound, and $c = \inf(B)$ exists in D with $c \geqslant x$. Since $\mathcal{M}, \tau, x \vDash \boxed{P}\varphi$ and $\mathcal{M}, \tau, x \vDash \boxed{P}\varphi \rightarrow \Diamond\boxed{P}\varphi$, we know $\mathcal{M}, \tau, x \vDash \Diamond\boxed{P}\varphi$, so there exists $x' > x$ with $\mathcal{M}, \tau, x' \vDash \boxed{P}\varphi$. Then $\mathcal{M}, \tau, y' \vDash \varphi$ for all $y' < x'$, so no element of B is less than x' and hence $c \geqslant x' > x$.

Let $y' < c$ be arbitrary. If $y' < x$, then $\mathcal{M}, \tau, y' \vDash \varphi$ by $\boxed{P}\varphi$. If $y' = x$, then $\mathcal{M}, \tau, y' \vDash \varphi$ since $x < x'$ and $\mathcal{M}, \tau, x' \vDash \boxed{P}\varphi$. If $x < y' < c$, it follows that $y' \notin B$, so $\mathcal{M}, \tau, y' \vDash \varphi$. Since $y' < c$ was arbitrary, we know that $\mathcal{M}, \tau, c \vDash \boxed{P}\varphi$. Given that $\mathcal{M}, \tau, c \vDash \boxed{P}\varphi \rightarrow \Diamond\boxed{P}\varphi$, we have $\mathcal{M}, \tau, c \vDash \Diamond\boxed{P}\varphi$, and so there exists $z > c$ with $\mathcal{M}, \tau, z \vDash \boxed{P}\varphi$. Since $c = \inf(B)$, there exists $b \in B$ with $c \leqslant b < z$. Then $\mathcal{M}, \tau, b \nvDash \varphi$ with $b < z$, contradicting $\mathcal{M}, \tau, z \vDash \boxed{P}\varphi$. Therefore $\mathcal{M}, \tau, x \vDash \boxed{F}\varphi$. Since $\mathcal{F}$, $\mathcal{M}$, τ, and x were arbitrary, we may conclude that $\triangle(\boxed{P}\varphi \rightarrow \Diamond\boxed{P}\varphi) \rightarrow (\boxed{P}\varphi \rightarrow \boxed{F}\varphi)$ is valid over every frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$.

($\Rightarrow$) Suppose $\mathcal{D}$ is not Complete. Then there exists a nonempty $S \subseteq D$ bounded above with no least upper bound. Let $L = \{x \in D : x < y \text{ for some } y \in S\}$ and $U = \{x \in D : y \leqslant x \text{ for all } y \in S\}$. Since $\mathcal{D}$ is totally ordered, L and U partition D with every element of L strictly less than every element of U. Since S has no least upper bound, L has no maximum and U has no minimum. Let $W = D$ with $w \Rightarrow_d u$ *iff* $u = w + d$ for $d \geqslant 0$, let $\mathcal{F}' = \langle W, \mathcal{D}, \Rightarrow \rangle$, and define $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ with $|p| = L$ and $\tau : D \rightarrow W$ by $\tau(x) = x$. As in **T12**, the frame $\mathcal{F}'$ satisfies both directions of *Compositionality*, together with *Seriality*, *Limit*, and *Spherical*, where $\tau \in H_{\mathcal{F}'}$ is total.



Given **D26**, it follows that $\mathcal{M}, \tau, x \vDash p$ if and only if $x \in L$. Pick any $x \in L$. Since L is downward closed, $\mathcal{M}, \tau, x \vDash \boxed{P}p$. Since U is nonempty and every $y \in U$ satisfies $y > x$ and $\tau(y) = y \notin |p|$, we have $\mathcal{M}, \tau, x \nvDash \boxed{F}p$.

It remains to verify $\mathcal{M}, \tau, x \vDash \triangle(\boxed{P}p \rightarrow \Diamond\boxed{P}p)$. Let $y \in D$ be arbitrary. If $y \in U$, then since U has no minimum, there exists $y' \in U$ with $y' < y$, and $\tau(y') = y' \notin |p|$, so $\mathcal{M}, \tau, y \nvDash \boxed{P}p$ and the conditional holds vacuously. If $y \in L$, then $\mathcal{M}, \tau, y \vDash \boxed{P}p$ since L is downward closed. Since L has no maximum, there exists $y' \in L$ with $y' > y$. Every $z < y'$ lies in L by downward closure, so $\mathcal{M}, \tau, y' \vDash \boxed{P}p$, giving $\mathcal{M}, \tau, y \vDash \Diamond\boxed{P}p$. Since y was arbitrary, $\mathcal{M}, \tau, x \vDash \triangle(\boxed{P}p \rightarrow \Diamond\boxed{P}p)$. Since $\mathcal{M}, \tau, x \vDash \boxed{P}p$ but $\mathcal{M}, \tau, x \nvDash \boxed{F}p$, we have $\mathcal{M}, \tau, x \nvDash \boxed{P}p \rightarrow \boxed{F}p$, and so $\nvDash_{\mathcal{F}'} \triangle(\boxed{P}\varphi \rightarrow \Diamond\boxed{P}\varphi) \rightarrow (\boxed{P}\varphi \rightarrow \boxed{F}\varphi)$, and the axiom is not valid over every frame with temporal order $\mathcal{D}$. $\square$

L16 *If $\mathcal{F}$ is* Deterministic, *then $\langle\tau\rangle_x = \{\tau\}$ for all $\tau \in H_{\mathcal{F}}$ and $x \in D$.*

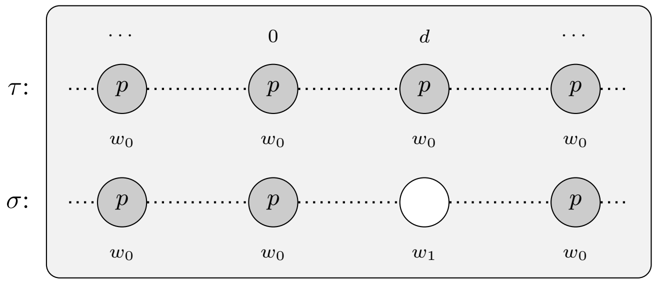
Proof. Suppose $\mathcal{F}$ is Deterministic. Since $\tau(x) = \tau(x)$, we have $\tau \in \langle\tau\rangle_x$. For the reverse inclusion, let $\sigma \in \langle\tau\rangle_x$ so that $\sigma(x) = \tau(x)$. For any $y \in D$, since τ is a possible world, $\tau(x) \Rightarrow_{y-x} \tau(y)$ by **D24**. Since σ is a possible world, $\sigma(x) \Rightarrow_{y-x} \sigma(y)$ by **D24**, and so $\tau(x) \Rightarrow_{y-x} \sigma(y)$. By Deterministic (**D28**), $\tau(y) = \sigma(y)$. Since y was arbitrary, we may conclude that $\sigma = \tau$ as desired. □

T15 $\mathcal{F} \vDash \varphi \to \boxdot\varphi$ *if $\mathcal{F}$ is* Deterministic.

Proof. Let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ be Deterministic. Let $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ be a model over $\mathcal{F}$ with $\tau \in H_{\mathcal{F}}$ and $x \in D$ where $\mathcal{M}, \tau, x \vDash \varphi$. By **L16**, $\langle\tau\rangle_x = \{\tau\}$, so $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in \langle\tau\rangle_x$. Therefore $\mathcal{M}, \tau, x \vDash \boxdot\varphi$. Since $\mathcal{M}$, τ, and x were chosen arbitrarily, $\vDash_{\mathcal{F}} \varphi \to \boxdot\varphi$ for every deterministic frame $\mathcal{F}$. □

T16 $\mathcal{F}' \nvDash \varphi \to \boxdot\varphi$ *for some non-deterministic frame $\mathcal{F}'$.*

Proof. Let $\mathcal{D}$ be a discrete temporal order with least positive duration d. Letting $W = \{w_0, w_1\}$ where $w \Rightarrow_0 w'$ *iff* $w = w'$ and $w \Rightarrow_z w'$ for all $w, w' \in W$ and $z > 0$, let $\mathcal{F}' = \langle W, \mathcal{D}, \Rightarrow\rangle$, and define $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ with $|p| = \{w_0\}$ and $\tau, \sigma : D \to W$ by $\tau(y) = w_0$ for all $y \in D$, and $\sigma(y) = w_1$ if $y = d$ and $\sigma(y) = w_0$ otherwise. Both directions of *Compositionality* are immediate, taking $u = w$ or $u = w'$ for the factoring direction when a summand is zero and any $u \in W$ otherwise, *Seriality* holds since every fiber at a positive duration is all of W while every state loops at duration zero, and $(w)_d = \{w\}$ gives $\bigcap_{z>0}(w)_z = \{w\}$ for *Limit*. *Spherical* holds by **C4** since W is finite, while every function from D to W is a total world history over $\mathcal{F}'$ since every non-zero difference between times is covered by the relation. Since $w_0 \Rightarrow_d w_0$ and $w_0 \Rightarrow_d w_1$, the frame $\mathcal{F}'$ is not Deterministic, and both τ and σ are in $H_{\mathcal{F}'}$.



Since $\sigma(0) = w_0 = \tau(0)$, we have $\sigma \in \langle\tau\rangle_0$. Since $\tau(y) = w_0 \in |p|$ for all $y > 0$, we have $\mathcal{M}, \tau, 0 \vDash \boxed{F}p$ by **D26**. However, $\sigma(d) = w_1 \notin |p|$ and $d > 0$, so $\mathcal{M}, \sigma, 0 \nvDash \boxed{F}p$ by **D26**. Since $\sigma \in \langle\tau\rangle_0$ and $\mathcal{M}, \sigma, 0 \nvDash \boxed{F}p$, we have $\mathcal{M}, \tau, 0 \nvDash \boxdot\boxed{F}p$. Thus we may conclude that $\mathcal{M}, \tau, 0 \nvDash \boxed{F}p \to \boxdot\boxed{F}p$, and so $\nvDash_{\mathcal{F}'} \varphi \to \boxdot\varphi$ more generally, giving a non-deterministic frame over which the schema is invalid. □

D30 The language $\mathcal{L}^+ := \langle \mathbb{L}, \bot, \to, \Box, \lhd, \rhd \rangle$ where $\mathbb{L} := \{p_i : i \in \mathbb{N}\}$ is a countable set of sentence letters as before where the remaining symbols denote falsity, material implication, the metaphysical necessity operator, the since operator, and the until operator, respectively. Well-formed sentences of $\mathcal{L}^+$ are defined by:

$$\varphi, \psi ::= p_i \mid \bot \mid \varphi \to \psi \mid \Box\varphi \mid \varphi \lhd \psi \mid \varphi \rhd \psi.$$

D31 The *models* of $\mathcal{L}^+$ are defined as in **D26**, where *truth in a model* $\mathcal{M}$ at $\tau \in H_{\mathcal{F}}$ and $x \in D$ extends the semantics **D26** with the following clauses:

($\lhd$) $\mathcal{M}, \tau, x \models \varphi \lhd \psi$ *iff* $\mathcal{M}, \tau, z \models \psi$ for some time $z < x$ where $\mathcal{M}, \tau, y \models \varphi$ for all $y \in D$ with $z < y < x$.

($\rhd$) $\mathcal{M}, \tau, x \models \varphi \rhd \psi$ *iff* $\mathcal{M}, \tau, z \models \psi$ for some time $z > x$ where $\mathcal{M}, \tau, y \models \varphi$ for all $y \in D$ with $x < y < z$.

D32 The following operators are defined in $\mathcal{L}^+$:

Past: $\Diamond_{P}\varphi := \top \lhd \varphi$.

Future: $\Diamond_{F}\varphi := \top \rhd \varphi$.

Historical: $\boxed{\scriptstyle P}\varphi := \neg\Diamond_{P}\neg\varphi$.

Henceforth: $\boxed{\scriptstyle F}\varphi := \neg\Diamond_{F}\neg\varphi$.

Always: $\triangle\varphi := \boxed{\scriptstyle P}\varphi \wedge \varphi \wedge \boxed{\scriptstyle F}\varphi$.

Sometimes: $\triangledown\varphi := \Diamond_{P}\varphi \vee \varphi \vee \Diamond_{F}\varphi$.

Next: $\textcircled{\scriptsize F}\varphi := \bot \rhd \varphi$.

Previous: $\textcircled{\scriptsize P}\varphi := \bot \lhd \varphi$.

T17 *$\mathcal{M}, \tau, x \models \boxed{\scriptstyle P}\varphi$ iff $\mathcal{M}, \tau, y \models \varphi$ for all $y \in D$ with $y < x$; and $\mathcal{M}, \tau, x \models \boxed{\scriptstyle F}\varphi$ iff $\mathcal{M}, \tau, y \models \varphi$ for all $y \in D$ with $x < y$.*

Proof. By **D32**, $\boxed{\scriptstyle P}\varphi := \neg\Diamond_{P}\neg\varphi = \neg(\top \lhd \neg\varphi)$. By **D31**, $\mathcal{M}, \tau, x \models \top \lhd \neg\varphi$ iff there exists $z < x$ such that $\mathcal{M}, \tau, z \models \neg\varphi$ and $\mathcal{M}, \tau, y \models \top$ for all $y \in D$ with $z < y < x$. ($\Leftarrow$) If φ holds at every $y < x$, then no open interval (z, x) is throughout $\neg\varphi$, so $\top \lhd \neg\varphi$ is false, and so $\boxed{\scriptstyle P}\varphi$ holds. ($\Rightarrow$) Contrapositive: if φ fails at some $z' < x$, then $\neg\varphi$ holds at z', where $\top$ holds at all $y' \in D$ whatsoever, and so *a fortiori* whenever $z' < y' < x$. Thus $\mathcal{M}, \tau, x \models \top \lhd \neg\varphi$ by **D31**, and so $\boxed{\scriptstyle P}\varphi$ then fails at x by **D32**.

By **D32**, $\boxed{\scriptstyle F}\varphi := \neg\Diamond_{F}\neg\varphi = \neg(\top \rhd \neg\varphi)$. By **D31**, $\mathcal{M}, \tau, x \models \top \rhd \neg\varphi$ iff there exists $z > x$ such that $\mathcal{M}, \tau, z \models \neg\varphi$ and $\mathcal{M}, \tau, y \models \top$ for all $y \in D$ with $x < y < z$. ($\Leftarrow$) If φ holds at every $y > x$, then no open interval (x, z) is throughout $\neg\varphi$, so $\top \rhd \neg\varphi$ is false, and so $\boxed{\scriptstyle F}\varphi$ holds. ($\Rightarrow$) Contrapositive: if φ fails at some $z' > x$, then $\neg\varphi$ holds at z', where $\top$ holds at all $y' \in D$ whatsoever, and so *a fortiori* whenever $x < y' < z'$. Thus $\mathcal{M}, \tau, x \models \top \rhd \neg\varphi$ by **D31**, and so $\boxed{\scriptstyle F}\varphi$ then fails at x by **D32**. $\square$

T18 *Over* Discrete *frames, $\mathcal{M}, \tau, x \models \textcircled{\scriptsize F}\varphi$ iff $\mathcal{M}, \tau, y \models \varphi$ for the immediate successor y of x; and $\mathcal{M}, \tau, x \models \textcircled{\scriptsize F}\varphi \leftrightarrow \bot$ when x has no immediate successor. The past dual holds for $\textcircled{\scriptsize P}\varphi$.*

Proof. By **D32**, $\textcircled{\tiny F}\varphi \coloneqq \perp \rhd \varphi$. By **D31**, $\mathcal{M}, \tau, x \vDash \perp \rhd \varphi$ iff there exists $z > x$ with $\mathcal{M}, \tau, z \vDash \varphi$ and $\mathcal{M}, \tau, y \vDash \perp$ for all $y \in D$ with $x < y < z$. Since $\perp$ is false everywhere, the interval (x, z) must contain no elements of D. Over DISCRETE frames (**D28**), the interval (x, z) is empty precisely when z is the immediate successor of x. Hence $\mathcal{M}, \tau, x \vDash \textcircled{\tiny F}\varphi$ iff φ holds at the immediate successor of x; when no immediate successor exists, the required empty interval cannot be witnessed and $\textcircled{\tiny F}\varphi$ is equivalent to $\perp$. The proof for $\textcircled{\tiny P}\varphi \coloneqq \perp \lhd \varphi$ is similar. □

T19 *There is a non-deterministic frame over which $\varphi \to \boxdot\varphi$ is valid for every sentence φ of $\mathcal{L}^+$ extended with $\boxdot$.*

Proof. Let $\mathcal{F}^\circ = \langle W, \mathcal{D}, \Rightarrow \rangle$ where $W = \mathbb{R}$, $\mathcal{D} = \langle \mathbb{R}, +, 0, \leqslant \rangle$, and $w \Rightarrow_x u$ just in case $x \leqslant u - w \leqslant 2x$ for $x \geqslant 0$, extended to negative durations by **D20**, so that each world state drifts rightward along $\mathbb{R}$ at a rate nondeterministically between 1 and 2. Since $\mathtt{fib}(w, x) = w + [x, 2x]$ for $x \geqslant 0$, the displacements composable over durations x and y form the interval sum $[x, 2x] + [y, 2y] \coloneqq \{a + b : a \in [x, 2x],\ b \in [y, 2y]\}$, which is $[x + y, 2(x + y)]$ since intervals are convex. For the composing direction of *Compositionality*, if $w \Rightarrow_x u$ and $u \Rightarrow_y v$, then $v - w = (u - w) + (v - u) \in [x, 2x] + [y, 2y] = [x + y, 2(x + y)]$, and so $w \Rightarrow_{x+y} v$. For the factoring direction, if $w \Rightarrow_{x+y} v$ where $x + y > 0$, then $\lambda \coloneqq (v - w)/(x + y) \in [1, 2]$ and $u \coloneqq w + \lambda x$ satisfies $w \Rightarrow_x u$ and $u \Rightarrow_y v$ by travelling at the constant rate λ across both legs, while $x + y = 0$ forces $x = y = 0$ and $v = w$, where $u \coloneqq w$ suffices. *Seriality* holds since $w \Rightarrow_x (w + x)$ and $(w - x) \Rightarrow_x w$ for every $w \in W$ and $x \geqslant 0$. For *Limit*, $\mathtt{fib}(w, y) = [w + y, w + 2y]$ for $0 \leqslant y < x$ and $\mathtt{fib}(w, y) = [w + 2y, w + y]$ for $-x < y < 0$, so $(w)_x \subseteq (w - 2x, w + 2x)$, while $w \in \mathtt{fib}(w, 0) \subseteq (w)_x$ for every $x > 0$, whence $\bigcap_{x>0}(w)_x = \{w\}$. For *Spherical*, every fiber is a compact interval of $\mathbb{R}$ and every segment is an intersection of two fibers, hence compact, so a directed family of nonempty fibers and segments has the finite intersection property, and its intersection is nonempty by the Heine–Borel theorem. Thus $\mathcal{F}^\circ$ is a frame despite its non-determinism.

By **D24**, the possible worlds in $H_{\mathcal{F}^\circ}$ are exactly the functions $\tau : \mathbb{R} \to \mathbb{R}$ where $\tau(y) - \tau(x) \in [y - x, 2(y - x)]$ for all $x < y$. Each possible world is therefore strictly increasing and 2-Lipschitz hence continuous, and is unbounded in both directions since $\tau(t) \geqslant \tau(0) + t$ for $t > 0$ and $\tau(t) \leqslant \tau(0) + t$ for $t < 0$, so every possible world is an order-isomorphism of $(\mathbb{R}, <)$ which passes through every world state exactly once. Moreover, for any $w \in W$ and $x \in D$, the translate $\delta(t) \coloneqq t + w - x$ is a possible world with $\delta(x) = w$, so every world state occurs at every time in some possible world.

I claim that for every well-formed φ and model $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ over $\mathcal{F}^\circ$ there is a set $S_\varphi \subseteq W$ where $\mathcal{M}, \tau, x \vDash \varphi$ *iff* $\tau(x) \in S_\varphi$ for all $\tau \in H_{\mathcal{F}^\circ}$ and $x \in D$, arguing by induction on the complexity of φ. For the base cases, $S_{p_i} = |p_i|$ and $S_\perp = \varnothing$ by **D26**, and $S_{\varphi \to \psi} = (W \backslash S_\varphi) \cup S_\psi$ follows from the induction hypothesis. Since every possible world is a strictly increasing continuous unbounded bijection, $\{\tau(y) : y > x\} = (\tau(x), \infty)$, so $\mathcal{M}, \tau, x \vDash \boxed{\tiny F}\varphi$ *iff* $\tau(y) \in S_\varphi$ for all $y > x$ by the induction hypothesis *iff* $(\tau(x), \infty) \subseteq S_\varphi$, whence $S_{\boxed{\tiny F}\varphi} = \{w \in W : (w, \infty) \subseteq S_\varphi\}$, and symmetrically $S_{\boxed{\tiny P}\varphi} = \{w \in W : (-\infty, w) \subseteq S_\varphi\}$. Since each possible world restricts to an order-isomorphism of (x, ∞) onto $(\tau(x), \infty)$ and of $(-\infty, x)$ onto $(-\infty, \tau(x))$, the times $z > x$ and intermediate times $x < y < z$ correspond to the world states $v > \tau(x)$ and intermediate states $\tau(x) < u < v$, whence $S_{\varphi \rhd \psi} = \{w \in W : v \in S_\psi$ and $(w, v) \subseteq S_\varphi$ for some $v > w\}$ by the induction hypothesis, and symmetrically $S_{\varphi \lhd \psi} = \{w \in W : v \in S_\psi$ and $(v, w) \subseteq S_\varphi$ for some $v < w\}$. Since every world state occurs at every time in some possible world, $\{\rho(x) : \rho \in H_{\mathcal{F}^\circ}\} = W$, so $\mathcal{M}, \tau, x \vDash \Box\varphi$ *iff* $\rho(x) \in S_\varphi$ for all $\rho \in H_{\mathcal{F}^\circ}$ by **D26** and the induction hypothesis *iff* $S_\varphi = W$, whence $S_{\Box\varphi} = W$ if $S_\varphi = W$ and $S_{\Box\varphi} = \varnothing$ otherwise. Finally, $\mathcal{M}, \tau, x \vDash \boxdot\varphi$ *iff* $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in \langle\tau\rangle_x$ *iff* $\sigma(x) \in S_\varphi$ whenever $\sigma(x) = \tau(x)$ by the induction hypothesis *iff* $\tau(x) \in S_\varphi$, whence $S_{\boxdot\varphi} = S_\varphi$, completing the induction.

Now suppose $\mathcal{M}, \tau, x \vDash \varphi$ for a model $\mathcal{M}$ over $\mathcal{F}^\circ$, possible world $\tau \in H_{\mathcal{F}^\circ}$, and time $x \in D$, so that $\tau(x) \in S_\varphi$. Every $\sigma \in \langle\tau\rangle_x$ satisfies $\sigma(x) = \tau(x) \in S_\varphi$, and so $\mathcal{M}, \sigma, x \vDash \varphi$. Therefore $\mathcal{M}, \tau, x \vDash \boxdot\varphi$, and since $\mathcal{M}$, τ, and x were arbitrary, $\vDash_{\mathcal{F}^\circ} \varphi \to \boxdot\varphi$ by **D29**. However, $0 \Rightarrow_1 1$ and $0 \Rightarrow_1 2$ where $1 \neq 2$, so $\mathcal{F}^\circ$ is not Deterministic by **D28**, giving a non-deterministic frame over which $\varphi \to \boxdot\varphi$ is valid.

The restriction to sentences without store and recall operators is essential. Letting $|p| = [3/2, \infty)$, the possible worlds $\tau(t) \coloneqq t$ and $\sigma(t) \coloneqq 2t$ satisfy $\sigma(0) = 0 = \tau(0)$, so $\tau \in \langle\sigma\rangle_0$, while $\sigma(1) = 2 \in |p|$ and $\tau(1) = 1 \notin |p|$. Thus $\mathcal{M}, \sigma, 0, \vec{v}_{[1/v_2]} \vDash \downarrow^2_{\mathrm{T}} p$ but $\mathcal{M}, \sigma, 0, \vec{v}_{[1/v_2]} \nvDash \boxdot\downarrow^2_{\mathrm{T}} p$, witnessed by τ, and so $\mathcal{F}^\circ$ refutes `Det` while validating *Determined* over the operator language. $\square$

T20 $\uparrow^1_{\mathrm{T}}\boxed{\mathrm{F}}\uparrow^2_{\mathrm{T}}\downarrow^1_{\mathrm{T}}(\boxdot\downarrow^2_{\mathrm{T}}\neg\varphi \vee \boxdot\downarrow^2_{\mathrm{T}}\varphi)$ *is valid over every deterministic frame, and invalid over some non-deterministic frame.*

Proof. We begin by observing that the biconditionals below follow from **D26**:

$$\begin{aligned} &\mathcal{M}, \tau, x, \vec{v} \vDash \uparrow^1_{\mathrm{T}}\boxed{\mathrm{F}}\uparrow^2_{\mathrm{T}}\downarrow^1_{\mathrm{T}}(\boxdot\downarrow^2_{\mathrm{T}}\neg\varphi \vee \boxdot\downarrow^2_{\mathrm{T}}\varphi) && (*)\\ &\quad\Leftrightarrow\ \mathcal{M}, \tau, x, \vec{v}_{[x/v_1]} \vDash \boxed{\mathrm{F}}\uparrow^2_{\mathrm{T}}\downarrow^1_{\mathrm{T}}(\boxdot\downarrow^2_{\mathrm{T}}\neg\varphi \vee \boxdot\downarrow^2_{\mathrm{T}}\varphi)\\ &\quad\Leftrightarrow\ \mathcal{M}, \tau, y, \vec{v}_{[x/v_1]} \vDash \uparrow^2_{\mathrm{T}}\downarrow^1_{\mathrm{T}}(\boxdot\downarrow^2_{\mathrm{T}}\neg\varphi \vee \boxdot\downarrow^2_{\mathrm{T}}\varphi) \text{ for all } y > x\\ &\quad\Leftrightarrow\ \mathcal{M}, \tau, y, \vec{v}_{[x/v_1][y/v_2]} \vDash \downarrow^1_{\mathrm{T}}(\boxdot\downarrow^2_{\mathrm{T}}\neg\varphi \vee \boxdot\downarrow^2_{\mathrm{T}}\varphi) \text{ for all } y > x\\ &\quad\Leftrightarrow\ \mathcal{M}, \tau, x, \vec{v}_{[x/v_1][y/v_2]} \vDash \boxdot\downarrow^2_{\mathrm{T}}\neg\varphi \vee \boxdot\downarrow^2_{\mathrm{T}}\varphi \text{ for all } y > x. \end{aligned}$$

Let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ be Deterministic. Choose an arbitrary world $\tau \in H_{\mathcal{F}}$, times $x, y \in D$ where $x < y$, and stored times $\vec{v}$ in a model $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ over $\mathcal{F}$. By **L16**, it follows that $\langle\tau\rangle_x = \{\tau\}$. If $\mathcal{M}, \tau, y, \vec{v}_{[x/v_1][y/v_2]} \vDash \varphi$, then $\mathcal{M}, \tau, x, \vec{v}_{[x/v_1][y/v_2]} \vDash \boxdot\downarrow^2_{\mathrm{T}}\varphi$ by **D26** and **T15**. If $\mathcal{M}, \tau, y, \vec{v}_{[x/v_1][y/v_2]} \nvDash \varphi$, then $\mathcal{M}, \tau, x, \vec{v}_{[x/v_1][y/v_2]} \vDash \boxdot\downarrow^2_{\mathrm{T}}\neg\varphi$ by the same reasoning. Thus $\mathcal{M}, \tau, x, \vec{v}_{[x/v_1][y/v_2]} \vDash \boxdot\downarrow^2_{\mathrm{T}}\neg\varphi \vee \boxdot\downarrow^2_{\mathrm{T}}\varphi$ by **D26**. Given the biconditionals $(*)$, we conclude this direction of the proof.

By the same non-deterministic frame $\mathcal{F}'$ and countermodel presented in **T16**, we have $\sigma(0) = w_0 = \tau(0)$ where $\sigma, \tau \in \langle\tau\rangle_0$. Since $\tau(d) = w_0 \in |p|$, we know that $\mathcal{M}, \tau, d, \vec{v}_{[0/v_1][d/v_2]} \vDash p$, and so we may draw the following conclusions by **D26**:

$$\mathcal{M}, \tau, d, \vec{v}_{[0/v_1][d/v_2]} \vDash p \ \Leftrightarrow\ \mathcal{M}, \tau, 0, \vec{v}_{[0/v_1][d/v_2]} \vDash \downarrow^2_{\mathrm{T}} p \tag{1}$$

$$\mathcal{M}, \tau, d, \vec{v}_{[0/v_1][d/v_2]} \nvDash \neg p \;\Leftrightarrow\; \mathcal{M}, \tau, 0, \vec{v}_{[0/v_1][d/v_2]} \nvDash \downarrow^2_{\mathrm{T}} \neg p \tag{2}$$

Since the conditions on the left are equivalent, all four sides of the biconditionals above hold. Given that $\sigma(d) = w_1 \notin |p|$ we know that $\mathcal{M}, \sigma, d, \vec{v}_{[0/v_1][d/v_2]} \nvDash p$ and so we may draw the following conclusions where again all four conditions are equivalent:

$$\mathcal{M}, \sigma, d, \vec{v}_{[0/v_1][d/v_2]} \nvDash p \;\Leftrightarrow\; \mathcal{M}, \sigma, 0, \vec{v}_{[0/v_1][d/v_2]} \nvDash \downarrow^2_{\mathrm{T}} p \tag{3}$$
$$\mathcal{M}, \sigma, d, \vec{v}_{[0/v_1][d/v_2]} \vDash \neg p \;\Leftrightarrow\; \mathcal{M}, \sigma, 0, \vec{v}_{[0/v_1][d/v_2]} \vDash \downarrow^2_{\mathrm{T}} \neg p \tag{4}$$

Since $\tau, \sigma \in \langle \tau \rangle_0$, it follows from (3) that $\mathcal{M}, \tau, 0, \vec{v}_{[0/v_1][d/v_2]} \nvDash \boxdot\downarrow^2_{\mathrm{T}} p$, witnessed by σ, and it follows from (2) that $\mathcal{M}, \tau, 0, \vec{v}_{[0/v_1][d/v_2]} \nvDash \boxdot\downarrow^2_{\mathrm{T}} \neg p$, witnessed by τ itself. Thus $\mathcal{M}, \tau, 0, \vec{v}_{[0/v_1][d/v_2]} \nvDash \boxdot\downarrow^2_{\mathrm{T}} \neg p \vee \boxdot\downarrow^2_{\mathrm{T}} p$. By $(*)$ above, we may conclude the proof. □

5.4 Soundness

The purely modal axioms **MK**, **MT**, and **M5** are sound by standard reasoning since $\Box$ quantifies universally over all possible worlds $H_{\mathcal{F}}$ without accessibility restriction, validating an S5 modal logic. Similarly, the temporal axioms **TK**, **TL**, **T4**, **TA**, and **TB** are sound given that $\mathcal{D}$ is a totally ordered abelian group where $\mathcal{D}$ is unbounded. Concise semantic proofs for representative axioms from each group are provided below, and the full soundness proof has been formalized in the Lean 4 repository for this paper. I will nevertheless present validity proofs for a number of characteristic axioms including the bimodal interaction axiom **MF** and the exchange rule **TD**.

D33 A conclusion φ is a *logical consequence* of a set of premises Γ— written $\Gamma \vDash \varphi$— just in case for all models $\mathcal{M}$, possible worlds $\tau \in H_{\mathcal{F}}$, and times $x \in D$, if $\mathcal{M}, \tau, x \vDash \gamma$ for all premises $\gamma \in \Gamma$, then $\mathcal{M}, \tau, x \vDash \varphi$. A sentence φ is *valid* just in case $\vDash \varphi$.

D34 The derivation relation $\vdash$ for **TM** is the smallest relation closed under the axioms and rules for **TM** as presented in **§3.2**.

D35 The proof system **TM** is *sound* with respect to the task semantics just in case for every sentence φ and set of sentences Γ in $\mathcal{L}$, $\Gamma \vDash \varphi$ whenever $\Gamma \vdash \varphi$.

T21 $\vDash \Diamond\Box\varphi \to \Box\varphi$.

Proof. Suppose $\mathcal{M}, \tau, x \vDash \Diamond\Box\varphi$. By **D26**, there exists $\sigma \in H_{\mathcal{F}}$ such that $\mathcal{M}, \sigma, x \vDash \Box\varphi$. Then $\mathcal{M}, \rho, x \vDash \varphi$ for all $\rho \in H_{\mathcal{F}}$. But this is exactly the condition for $\mathcal{M}, \tau, x \vDash \Box\varphi$, since the quantification ranges over $H_{\mathcal{F}}$. Thus $\vDash \Diamond\Box\varphi \to \Box\varphi$. □

L17 *Let $n : D \to D$ be defined by $n(x) \coloneqq -x$, let $\mathcal{F}^- = \langle W, \mathcal{D}, \Rightarrow^- \rangle$ where $w \Rightarrow^-_x u \coloneqq w \Rightarrow_{-x} u$, and let $\tau^- = \tau \circ n$ for each $\tau \in H_{\mathcal{F}}$. Then $\mathcal{F}^-$ is a task frame and $\tau \mapsto \tau^-$ is a bijection $H_{\mathcal{F}} \to H_{\mathcal{F}^-}$, and for all well-formed sentences φ of $\mathcal{L}$, models $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ over $\mathcal{F}$ and $\mathcal{M}^- = \langle W, \mathcal{D}, \Rightarrow^-, |\cdot| \rangle$ over $\mathcal{F}^-$, possible worlds*

$\tau \in H_{\mathcal{F}}$*, and times* $x \in D$*:*

$$\mathcal{M}, \tau, x \vDash \varphi \;\Leftrightarrow\; \mathcal{M}^{-}, \tau^{-}, n(x) \vDash \varphi_{\langle \mathrm{P}|\mathrm{F} \rangle}.$$

Proof. Since $\mathcal{D}$ is a totally ordered abelian group, it follows that n is an order-reversing automorphism satisfying the following:

$$n(x+y) = -(x+y) = (-x) + (-y) = n(x) + n(y) \tag{†}$$
$$x < y \Leftrightarrow -y < -x \Leftrightarrow n(y) < n(x) \tag{‡}$$

On the positive cone, $\Rightarrow^{-}$ is the converse of $\Rightarrow$, and so the composing direction of *Compositionality* on D^{+} for $\Rightarrow^{-}$ is the converse of the composing direction for $\Rightarrow$, obtained by composing the converse witnesses in reverse order with $-x+-y = -(x+y)$ by (†). The factoring direction transfers by the mirror: given $w \Rightarrow^{-}_{x+y} v$ with $x, y \geqslant 0$ so that $v \Rightarrow_{x+y} w$, the factoring direction for $\Rightarrow$ with $x+y = y+x$ yields some $u \in W$ where $v \Rightarrow_{y} u$ and $u \Rightarrow_{x} w$, and so $w \Rightarrow^{-}_{x} u$ and $u \Rightarrow^{-}_{y} v$. *Seriality* is immediate since the forward fibers of $\Rightarrow^{-}$ are the backward fibers of $\Rightarrow$ and conversely, so all are nonempty. *Limit* also transfers since the cones of $\Rightarrow^{-}$ coincide with those of $\Rightarrow$, and *Spherical* transfers since the fibers and segments coincide.

Since n is a bijection of D, the function $\tau^{-} = \tau \circ n$ is total. Given $a, b \in D$, we have $\tau^{-}(a) \Rightarrow^{-}_{b-a} \tau^{-}(b)$ just in case $\tau(-a) \Rightarrow_{a-b} \tau(-b)$, which holds by **D24** applied to τ at $-a, -b \in D$, and so $\tau^{-} \in H_{\mathcal{F}^{-}}$. Since $n \circ n = \mathrm{id}$ so that $(\mathcal{F}^{-})^{-} = \mathcal{F}$, the same argument applied to $\mathcal{F}^{-}$ shows that $\sigma \in H_{\mathcal{F}^{-}}$ implies $\sigma^{-} \in H_{\mathcal{F}}$, and so $\tau \mapsto \tau^{-}$ is a bijection $H_{\mathcal{F}} \to H_{\mathcal{F}^{-}}$ with itself as inverse. The proof of the displayed biconditional is by induction on the complexity of φ.

Base Case (p_i): Since $(p_i)_{\langle \mathrm{P}|\mathrm{F} \rangle} = p_i$ and $\mathcal{M}, \mathcal{M}^{-}$ share the interpretation $|\cdot|$, the biconditional follows from **D26**:

$$\begin{aligned} \mathcal{M}, \tau, x \vDash p_i \;&\Leftrightarrow\; \tau(x) \in |p_i| \\ &\Leftrightarrow\; \tau^{-}(n(x)) \in |p_i| \\ &\Leftrightarrow\; \mathcal{M}^{-}, \tau^{-}, n(x) \vDash p_i \end{aligned}$$

The second biconditional holds since $\tau^{-}(n(x)) = \tau(n(n(x))) = \tau(x)$.

Base Case ($\bot$): $\mathcal{M}, \tau, x \nvDash \bot$ and $\mathcal{M}^{-}, \tau^{-}, n(x) \nvDash \bot$ by **D26**.

Inductive Case ($\varphi \to \psi$): Since $(\varphi \to \psi)_{\langle \mathrm{P}|\mathrm{F} \rangle} = \varphi_{\langle \mathrm{P}|\mathrm{F} \rangle} \to \psi_{\langle \mathrm{P}|\mathrm{F} \rangle}$, the result follows from the inductive hypothesis and **D26**.

Inductive Case ($\Box\varphi$): Since $(\Box\varphi)_{\langle \mathrm{P}|\mathrm{F} \rangle} = \Box\varphi_{\langle \mathrm{P}|\mathrm{F} \rangle}$, the biconditional follows by **D26**:

$$\begin{aligned} \mathcal{M}, \tau, x \vDash \Box\varphi \;&\Leftrightarrow\; \mathcal{M}, \sigma, x \vDash \varphi \text{ for all } \sigma \in H_{\mathcal{F}} \\ &\Leftrightarrow\; \mathcal{M}^{-}, \sigma^{-}, n(x) \vDash \varphi_{\langle \mathrm{P}|\mathrm{F} \rangle} \text{ for all } \sigma \in H_{\mathcal{F}} \\ &\Leftrightarrow\; \mathcal{M}^{-}, \rho, n(x) \vDash \varphi_{\langle \mathrm{P}|\mathrm{F} \rangle} \text{ for all } \rho \in H_{\mathcal{F}^{-}} \\ &\Leftrightarrow\; \mathcal{M}^{-}, \tau^{-}, n(x) \vDash \Box\varphi_{\langle \mathrm{P}|\mathrm{F} \rangle} \end{aligned}$$

The second biconditional holds by the inductive hypothesis, the third since $\sigma \mapsto \sigma^-$ is a bijection $H_{\mathcal{F}} \to H_{\mathcal{F}^-}$, and the fourth by **D26** applied to $\mathcal{M}^-$ over $\mathcal{F}^-$.

Inductive Case ($\boxed{\mathrm{F}}\varphi$): Since $(\boxed{\mathrm{F}}\varphi)_{\langle\mathrm{P}|\mathrm{F}\rangle} = \boxed{\mathrm{P}}\varphi_{\langle\mathrm{P}|\mathrm{F}\rangle}$, the biconditional follows by **D26**:

$$\begin{aligned}
\mathcal{M}, \tau, x \vDash \boxed{\mathrm{F}}\varphi \;&\Leftrightarrow\; \mathcal{M}, \tau, y \vDash \varphi \text{ for all } y \in D \text{ where } x < y \\
&\Leftrightarrow\; \mathcal{M}^-, \tau^-, n(y) \vDash \varphi_{\langle\mathrm{P}|\mathrm{F}\rangle} \text{ for all } y \in D \text{ where } x < y \\
&\Leftrightarrow\; \mathcal{M}^-, \tau^-, z \vDash \varphi_{\langle\mathrm{P}|\mathrm{F}\rangle} \text{ for all } z \in D \text{ where } z < n(x) \\
&\Leftrightarrow\; \mathcal{M}^-, \tau^-, n(x) \vDash \boxed{\mathrm{P}}\varphi_{\langle\mathrm{P}|\mathrm{F}\rangle}
\end{aligned}$$

The second biconditional holds by the inductive hypothesis. The third holds by (‡), and so substituting $z = n(y)$ transforms the domain of quantification.

Inductive Case ($\boxed{\mathrm{P}}\varphi$): By parity of reasoning with the $\boxed{\mathrm{F}}$ case, replacing $y > x$ with $y < x$ and applying (‡) in the opposite direction. □

T22 *If* $\vDash \varphi$*, then* $\vDash \varphi_{\langle\mathrm{P}|\mathrm{F}\rangle}$.

Proof. Suppose $\vDash \varphi$ and let $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ be a model of $\mathcal{L}$ defined over the task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$, where $\sigma \in H_{\mathcal{F}}$ and $y \in D$ be arbitrary. Let $\mathcal{F}^- = \langle W, \mathcal{D}, \Rightarrow^-\rangle$ and $\mathcal{M}^- = \langle W, \mathcal{D}, \Rightarrow^-, |\cdot|\rangle$ be as in **L17**. Since $\mathcal{F}^-$ is itself a task frame, $\mathcal{M}^-$ is a model of $\mathcal{L}$, and since $\vDash \varphi$ holds over every model, we have $\mathcal{M}^-, \sigma^-, n(y) \vDash \varphi$, where $\sigma^- \in H_{\mathcal{F}^-}$ by the bijection established in **L17**.

Applying **L17** to $\mathcal{M}^-$ over $\mathcal{F}^-$ with $\tau = \sigma^-$ and $x = n(y)$ yields:

$$(\mathcal{M}^-)^-, (\sigma^-)^-, n(n(y)) \vDash \varphi_{\langle\mathrm{P}|\mathrm{F}\rangle}.$$

Since $(\mathcal{F}^-)^- = \mathcal{F}$ so that $(\mathcal{M}^-)^- = \mathcal{M}$, and since $(\sigma^-)^- = \sigma$ and $n(n(y)) = y$, we conclude $\mathcal{M}, \sigma, y \vDash \varphi_{\langle\mathrm{P}|\mathrm{F}\rangle}$ as desired. Equivalently, $\vDash \varphi_{\langle\mathrm{P}|\mathrm{F}\rangle}$. □

T23 $\vDash \varphi \to \boxed{\mathrm{F}}\Diamond\varphi$.

Proof. Suppose $\mathcal{M}, \tau, x \vDash \varphi$. To show $\mathcal{M}, \tau, x \vDash \boxed{\mathrm{F}}\Diamond\varphi$, let $y > x$ be arbitrary. Thus $\mathcal{M}, \tau, y \vDash \Diamond\varphi$. Since $y > x$ was arbitrary, $\mathcal{M}, \tau, x \vDash \boxed{\mathrm{F}}\Diamond\varphi$. Thus $\vDash \varphi \to \boxed{\mathrm{F}}\Diamond\varphi$. □

T24 $\vDash (\Diamond\varphi \wedge \Diamond\psi) \to [\Diamond(\Diamond\varphi \wedge \psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\varphi \wedge \Diamond\psi)]$.

Proof. Suppose $\mathcal{M}, \tau, x \vDash \Diamond\varphi$ and $\mathcal{M}, \tau, x \vDash \Diamond\psi$. By **D26**, there exist $y > x$ and $z > x$ with $\mathcal{M}, \tau, y \vDash \varphi$ and $\mathcal{M}, \tau, z \vDash \psi$. Since $\mathcal{D}$ is totally ordered, either $y < z$, $y = z$, or $z < y$. If $y < z$, then $\mathcal{M}, \tau, y \vDash \varphi \wedge \Diamond\psi$, giving $\mathcal{M}, \tau, x \vDash \Diamond(\varphi \wedge \Diamond\psi)$. If $y = z$, then $\mathcal{M}, \tau, y \vDash \varphi \wedge \psi$, so $\mathcal{M}, \tau, x \vDash \Diamond(\varphi \wedge \psi)$. If $z < y$, then $\mathcal{M}, \tau, z \vDash \Diamond\varphi \wedge \psi$, giving $\mathcal{M}, \tau, x \vDash \Diamond(\Diamond\varphi \wedge \psi)$. Thus $\vDash (\Diamond\varphi \wedge \Diamond\psi) \to [\Diamond(\Diamond\varphi \wedge \psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\varphi \wedge \Diamond\psi)]$. □

T25 $\vDash \Box\varphi \to \Box\boxed{\mathrm{F}}\varphi$.

Proof. Suppose $\mathcal{M}, \tau, x \vDash \Box\varphi$. By **D26**, $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in H_{\mathcal{F}}$. In order to show $\mathcal{M}, \tau, x \vDash \Box\boxed{\mathrm{F}}\varphi$, we must prove that $\mathcal{M}, \sigma, x \vDash \boxed{\mathrm{F}}\varphi$ for all $\sigma \in H_{\mathcal{F}}$.

Let $\sigma \in H_{\mathcal{F}}$ be arbitrary and consider any $y > x$. By **L14**, there is some $\rho \in H_{\mathcal{F}}$ such that $\sigma \approx_y^x \rho$ is witnessed by the time-shift function $\bar{a}(z) = z - y + x$. By **L15**, $\mathcal{M}, \sigma, y \vDash \varphi$ if and only if $\mathcal{M}, \rho, x \vDash \varphi$. Since $\mathcal{M}, \tau, x \vDash \Box\varphi$, we have $\mathcal{M}, \rho, x \vDash \varphi$. Therefore $\mathcal{M}, \sigma, y \vDash \varphi$ by time-shift invariance. Since $y > x$ was arbitrary, $\mathcal{M}, \sigma, x \vDash \boxed{\scriptstyle F}\varphi$. Since σ was arbitrary, $\mathcal{M}, \tau, x \vDash \Box\boxed{\scriptstyle F}\varphi$. Thus $\vDash \Box\varphi \to \Box\boxed{\scriptstyle F}\varphi$. □

T26 (Soundness) *If* $\vdash \varphi$, *then* $\vDash \varphi$.

Proof. By induction on derivations. The propositional tautologies are valid, and **MP** preserves validity by standard arguments. The S5 axioms **MK**, **MT**, and **M5** are valid given that $\Box$ quantifies universally over $H_{\mathcal{F}}$ and **MN** preserves validity. The temporal axioms **TK**, **TL**, **T4**, **TA**, and **TB** are valid given that $\mathcal{D}$ is a totally ordered abelian group, and **TD** preserves validity since there is an order-reversing automorphism $x \mapsto -x$ on $\mathcal{D}$. Representative proofs are given in **T21**, **T22**, **T23**, and **T24** above. The bimodal axiom **MF** is shown to be valid in **T25**. The proof of soundness otherwise proceeds as usual.[79] □

C6 *The perpetuity principles* ***P1*** *–* ***P6*** *are all valid.*

Proof. Follows immediately from **T26** given the derivations of **P1** – **P6** in §**3.2**. □

5.5 Proof Theory

This section presents the proof system $\mathbf{TM}^+$ for the extended language $\mathcal{L}^+$, establishing the completeness results of **C7**, implemented in the Lean 4 repository for this paper. All definitions are restated for convenience.

D36 The **S5** *Modal Logic* is the smallest extension of *Classical Propositional Logic* **CPL** closed under the following rule schemata and metarule:

MK	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.	**MP**	$\varphi,\ \varphi \to \psi \vdash \psi$.
MT	$\Box\varphi \to \varphi$.	**MN**	*If* $\vdash \varphi$, *then* $\vdash \Box\varphi$.
M5	$\Diamond\Box\varphi \to \Box\varphi$.		

D37 Letting $\varphi_{\langle \mathrm{S}|\mathrm{U}\rangle}$ denote the result swapping occurrences of $\triangleleft$ and $\triangleright$ in φ, **BX** is the *Base Burgess–Xu Tense Logic* axiomatized below where the past/since direction of each axiom follows from the future/until direction:

TN	*If* $\vdash \varphi$, *then* $\vdash \boxed{\scriptstyle F}\varphi$.	**TD**	*If* $\vdash \varphi$, *then* $\vdash \varphi_{\langle \mathrm{S}|\mathrm{U}\rangle}$.

The seriality **TB** and linearity **TL** axioms from **TM**, together with connectedness:

[79] The repository https://github.com/benbrastmckie/BimodalLogic for this paper implements the soundness theorem for **TM** in Lean 4, providing formal verification for this result.

TB $\langle F\rangle\top$.

TL $(\langle F\rangle\varphi \wedge \langle F\rangle\psi) \to [\langle F\rangle(\varphi \wedge \psi) \vee \langle F\rangle(\varphi \wedge \langle F\rangle\psi) \vee \langle F\rangle(\langle F\rangle\varphi \wedge \psi)]$.

CN $[(\varphi \rhd \psi) \wedge (\chi \rhd \theta)] \to [(\varphi \wedge \chi) \rhd (\psi \wedge \theta) \vee (\varphi \wedge \chi) \rhd (\psi \wedge \chi) \vee (\varphi \wedge \chi) \rhd (\varphi \wedge \theta)]$.

The primary axioms for $\lhd$ and $\rhd$ are:

TA $\varphi \to \boxed{F}\langle F\rangle\varphi$.

UE $(\varphi \rhd \psi) \to \langle F\rangle\psi$.

UT $\langle F\rangle\varphi \to (\top \rhd \varphi)$.

UI $\varphi \rhd (\varphi \wedge (\varphi \rhd \psi)) \to \varphi \rhd \psi$.

UC $\boxed{F}(\varphi \to \psi) \to ((\chi \rhd \varphi) \to (\chi \rhd \psi))$.

UF $(\varphi \rhd \psi) \to (\varphi \wedge (\varphi \rhd \psi)) \rhd \psi$.

UG $\boxed{F}(\varphi \to \chi) \to ((\varphi \rhd \psi) \to (\chi \rhd \psi))$.

SU $\theta \wedge (\varphi \rhd \psi) \to \varphi \rhd (\psi \wedge (\varphi \lhd \theta))$.

The uniformity axioms, which hold vacuously unless the order is discrete, are:

NP $\text{\textcircled{F}}\top \to \text{\textcircled{P}}\top$.

NF $\text{\textcircled{F}}\top \to \boxed{F}\text{\textcircled{F}}\top$.

NA $\text{\textcircled{F}}\top \to \boxed{P}\text{\textcircled{F}}\top$.

NB $\text{\textcircled{F}}\top \to \Box\text{\textcircled{F}}\top$.

The logic **BX** is smallest extension of **CPL** closed under all instances of the above.

D38 The *Discrete Burgess–Xu Tense Logic* $\mathbf{BX}_f$ is the smallest extension of the base logic **BX** to include all instances of the following axioms:

UZ $\langle F\rangle\varphi \to (\neg\varphi \rhd \varphi)$.

Z1 $\boxed{F}(\boxed{F}\varphi \to \varphi) \to (\langle F\rangle\boxed{F}\varphi \to \boxed{F}\varphi)$.

Whereas **UZ** asserts that if φ in the future, then there is a *nearest* future φ-time where $\neg\varphi$ throughout the intervening interval, **Z1** is a backward induction principle that is characteristic of successor-Archimedean frames. Since if follows by Hölder's theorem that a nontrivial *discrete* Archimedean totally ordered abelian group is isomorphic to $\mathbb{Z}$, the successor-Archimedean discrete class to which $\mathbf{BX}_f$ and $\mathbf{TM}^+_{\text{F}}$ are sound and complete is exactly $\mathbb{Z}$-time.

D39 The *Dense Burgess–Xu Tense Logic* $\mathbf{BX}_d$ is the smallest extension of the base logic **BX** to include all instances of the following axioms:

DN $\boxed{F}\boxed{F}\varphi \to \boxed{F}\varphi$.

NN $\neg\text{\textcircled{F}}\top$.

Whereas **DN** coincides with **DN** of **TM**, the axiom **NN** is specific to $\mathbf{TM}^+$ and asserts that there is no immediate successor.

D40 Letting $K^+\varphi \coloneqq \neg(\neg\varphi \rhd \top)$ and $K^-\varphi \coloneqq \neg(\neg\varphi \lhd \top)$ abbreviate Reynolds' (1992) operators— $K^+\varphi$ says that φ recurs arbitrarily soon in the future, and $K^-\varphi$ that φ recurred arbitrarily recently in the past— the *Complete Burgess–Xu Tense Logic* $\mathbf{BX}_c$ is the smallest extension of the base logic **BX** to include all instances of the following axioms, due to Reynolds (1992), where only the future/until direction of is stated, its past/since direction following by **TD**:

Prior-U $(\varphi \rhd \top) \wedge \langle\mathrm{F}\rangle\neg\varphi \to \varphi \rhd (\neg\varphi \vee K^{+}\neg\varphi)$.

Sep $K^{+}\varphi \wedge \neg K^{+}(\varphi \wedge (\neg\varphi \rhd \varphi)) \to K^{+}(K^{+}\varphi \wedge K^{-}\varphi)$.

The following axiom restates **CO** from **TM**, and is a *derived theorem* of $\mathbf{BX}_c$ rather than a further axiom, using only **Prior-U** and the base axioms of **BX**:

CO $\triangle(\boxed{\mathrm{P}}\varphi \to \langle\mathrm{F}\rangle\boxed{\mathrm{P}}\varphi) \to (\boxed{\mathrm{P}}\varphi \to \boxed{\mathrm{F}}\varphi)$.

As a result, **CO** may be omitted from $\mathbf{BX}_c$.

D41 The *Base Logic of Tense and Modality* $\mathbf{TM}^{+}$ for $\mathcal{L}^{+}$ is the smallest extension of **S5** and the base logic **BX** that includes the following *bimodal interaction* axiom:

MF $\Box\varphi \to \Box\boxed{\mathrm{F}}\varphi$.

Similarly, the discrete $\mathbf{TM}^{+}_{f}$, dense $\mathbf{TM}^{+}_{d}$, and complete $\mathbf{TM}^{+}_{c}$ extensions of $\mathbf{TM}^{+}$ include the additional axioms that distinguish $\mathbf{BX}_f$, $\mathbf{BX}_d$, and $\mathbf{BX}_c$, respectively.

C7 (Completeness) *Completeness is carried by the following $\mathcal{L}^{+}$ systems:*

$\mathbf{TM}^{+}$ *Strongly complete over all task frames.*

$\mathbf{TM}^{+}_{\mathbf{d}}$ *Strongly complete over the dense frames.*

$\mathbf{TM}^{+}_{\mathbf{f}}$ *Weakly complete over $\mathbb{Z}$-time.*

$\mathbf{TM}^{+}_{\mathbf{c}}$ *Weakly complete over the dense-and-complete class.*

Strong completeness provably fails for $\mathbb{Z}$-time as well as for the dense-and-complete class $\mathbb{R}$ where compactness fails, and so weak completeness is the appropriate target.

Proof. These results are machine-checked in the Lean 4 repository for this paper. □

Since derivations of **P1** – **P6** and **TF** are provided in §**3.2**, this section will conclude by deriving a number of additional interaction principles in $\mathbf{TM}^{+}$. I will begin by stating the following equivalences without proof.

P9 $\neg\triangle\varphi \leftrightarrow \triangledown\neg\varphi$. **P10** $\neg\triangledown\varphi \leftrightarrow \triangle\neg\varphi$.

Further derivations are provided in the Lean 4 repository for this paper.

P11 $\triangledown\Diamond\varphi \to \Diamond\varphi$.

Proof. Since $\triangledown\Diamond\varphi \coloneqq \langle\mathrm{P}\rangle\Diamond\varphi \vee \Diamond\varphi \vee \langle\mathrm{F}\rangle\Diamond\varphi$ where $\Diamond\varphi \to \Diamond\varphi$ is a theorem of classical propositional logic, it suffices to show $\langle\mathrm{F}\rangle\Diamond\varphi \to \Diamond\varphi$ and $\langle\mathrm{P}\rangle\Diamond\varphi \to \Diamond\varphi$.

By the derived theorem **TF** instantiated with $\neg\varphi$, we have $\Box\neg\varphi \to \boxed{\scriptstyle\mathrm{F}}\Box\neg\varphi$, so $\langle\mathrm{F}\rangle\Diamond\varphi \to \Diamond\varphi$ by contraposition. By **TD** applied to **TF**, we have $\Box\neg\varphi \to \boxed{\scriptstyle\mathrm{P}}\Box\neg\varphi$. It follows by contraposition that $\langle\mathrm{P}\rangle\Diamond\varphi \to \Diamond\varphi$ as desired.

P12 $\triangle\Diamond\varphi \to \Diamond\varphi$.

Proof. Since $\triangle\Diamond\varphi \coloneqq \boxed{\scriptstyle\mathrm{P}}\Diamond\varphi \wedge \Diamond\varphi \wedge \boxed{\scriptstyle\mathrm{F}}\Diamond\varphi$, conjunction elimination yields $\triangle\Diamond\varphi \to \Diamond\varphi$.

P13 $\triangledown\Box\varphi \leftrightarrow \Box\varphi$.

Proof. By **P6**, we have $\triangledown\Box\varphi \to \Box\triangle\varphi$. Since $\triangle\varphi \to \varphi$ by propositional logic, we know by **MN** and **MK** that $\Box\triangle\varphi \to \Box\varphi$. Therefore $\triangledown\Box\varphi \to \Box\varphi$. Since $\triangledown\varphi \coloneqq \langle\mathrm{P}\rangle\varphi \vee \varphi \vee \langle\mathrm{F}\rangle\varphi$, the reverse direction $\Box\varphi \to \triangledown\Box\varphi$ follows by disjunction introduction.

P14 $\triangle\Box\varphi \leftrightarrow \Box\varphi$.

Proof. Given the definition $\triangle\varphi \coloneqq \boxed{\scriptstyle\mathrm{P}}\varphi \wedge \varphi \wedge \boxed{\scriptstyle\mathrm{F}}\varphi$, the forward direction $\triangle\Box\varphi \to \Box\varphi$ follows by conjunction elimination. For the reverse direction, we have $\Box\varphi \to \Box\varphi$ trivially, $\Box\varphi \to \boxed{\scriptstyle\mathrm{F}}\Box\varphi$ by the derived theorem **TF**, and $\Box\varphi \to \boxed{\scriptstyle\mathrm{P}}\Box\varphi$ by **TD** applied to **TF**. Therefore $\Box\varphi \to (\boxed{\scriptstyle\mathrm{P}}\Box\varphi \wedge \Box\varphi \wedge \boxed{\scriptstyle\mathrm{F}}\Box\varphi)$, which is $\Box\varphi \to \triangle\Box\varphi$ as desired.

P15 $\Box\triangle\varphi \leftrightarrow \triangle\Box\varphi$.

Proof. Since $\triangle\varphi \to \varphi$ by definition, **MN** and **MK** yield $\Box\triangle\varphi \to \Box\varphi$. By **P14**, $\Box\varphi \to \triangle\Box\varphi$, and so $\Box\triangle\varphi \to \triangle\Box\varphi$, establishing the forward direction.

Since **P14** gives $\triangle\Box\varphi \to \Box\varphi$, we know $\Box\varphi \to \Box\triangle\varphi$ by **P3**, and so $\triangle\Box\varphi \to \Box\triangle\varphi$, establishing the reverse.

P16 $\Diamond\triangledown\varphi \leftrightarrow \triangledown\Diamond\varphi$.

Proof. By **P10**, $\neg\triangledown\varphi \leftrightarrow \triangle\neg\varphi$. From **MN** and **MK** we may obtain $\Box\neg\triangledown\varphi \leftrightarrow \Box\triangle\neg\varphi$, and so $\Diamond\triangledown\varphi \leftrightarrow \neg\Box\triangle\neg\varphi$. Instantiating **P15** with $\neg\varphi$ yields $\Box\triangle\neg\varphi \leftrightarrow \triangle\Box\neg\varphi$, and so $\neg\Box\triangle\neg\varphi \leftrightarrow \neg\triangle\Box\neg\varphi$. By instantiating **P9** with $\Box\neg\varphi$, we have $\neg\triangle\Box\neg\varphi \leftrightarrow \triangledown\neg\Box\neg\varphi$, and so $\neg\triangle\Box\neg\varphi \leftrightarrow \triangledown\Diamond\varphi$. Thus $\Diamond\triangledown\varphi \leftrightarrow \triangledown\Diamond\varphi$ by the transitivity of biconditionals.

P17 $\Diamond\triangledown\varphi \to \Diamond\varphi$.

Proof. Follows immediately from **P11** and **P16**.

P18 $\Box\triangle\varphi \leftrightarrow \Box\varphi$.

Proof. For the forward direction, since $\triangle\varphi \to \varphi$ by definition and propositional logic, **MN** and **MK** yield $\Box\triangle\varphi \to \Box\varphi$. Conversely, $\Box\varphi \to \triangle\varphi$ by **P1**, and so **MN** and **MK** yield $\Box\Box\varphi \to \Box\triangle\varphi$. Since $\Box\varphi \to \Box\Box\varphi$ by **MT**, **M5**, and **MK**, we obtain $\Box\varphi \to \Box\triangle\varphi$ as desired.

P19 $\Diamond\triangle\varphi \to \Diamond\varphi$.

Proof. Since $\triangle\varphi \to \varphi$ by propositional logic, we get $\neg\varphi \to \neg\triangle\varphi$ by contraposition. By **MN** and **MK**, we obtain $\Box\neg\varphi \to \Box\neg\triangle\varphi$, and so $\Diamond\triangle\varphi \to \Diamond\varphi$ by contraposition. □

P20 $\Diamond\varphi \leftrightarrow \Diamond\triangledown\varphi$.

Proof. Instantiating **P18** with $\neg\varphi$ yields $\Box\triangle\neg\varphi \leftrightarrow \Box\neg\varphi$. By **P10**, $\neg\triangledown\varphi \leftrightarrow \triangle\neg\varphi$, so from **MN** and **MK** we obtain $\Box\neg\triangledown\varphi \leftrightarrow \Box\triangle\neg\varphi$. Therefore $\Box\neg\triangledown\varphi \leftrightarrow \Box\neg\varphi$ by transitivity of biconditionals, and so $\Diamond\triangledown\varphi \leftrightarrow \Diamond\varphi$ by propositional logic. □

P21 $\triangledown\Diamond\varphi \to \Diamond\triangledown\varphi$.

Proof. Follows immediately from **P11** and **P20**. □

P22 $\Diamond\triangledown\varphi \to \triangledown\Diamond\varphi$.

Proof. By **P5**, we have $\Diamond\triangledown\varphi \to \triangle\Diamond\varphi$. Since $\triangle\Diamond\varphi \to \Diamond\varphi$ by conjunction elimination, and $\Diamond\varphi \to \triangledown\Diamond\varphi$ by definition, we obtain $\Diamond\triangledown\varphi \to \triangledown\Diamond\varphi$ as desired. □

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